CONCEPTUAL PROOFS OF THE MENGER AND ROTHBERGER GAMES

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Abstract. We provide conceptual proofs of the two most fundamental theorems concerning topological games and open covers: Hurewicz's Theorem concerning the Menger game, and Pawlikowski's Theorem concerning the Rothberger game.

1. Introduction

Topological games form a major tool in the study of topological properties and their relations to Ramsey theory, forcing, function spaces, and other related topics. Excellent surveys [11, 2], and a comprehensive bibliography for further inspection [3], are available. At the heart of the theory of *selection principles* [13], covering properties are defined by the ability to diagonalize, in canonical ways, sequences of open covers. Each of these covering properties has an associated two-player game. Often the nonexistence of a winning strategy for the first player in the associated game is equivalent to the original property, and this forms a strong tool for establishing results concerning the original property.

Allowing few potential exceptions, all proofs of results of this type use, as black boxes, two fundamental theorems. The original proofs of the fundamental theorems are technical and difficult to digest, and the lack of their understanding has undoubtedly veiled deep insights that necessitate variations in the original proofs. We present here new, conceptual proofs of these theorems. These proofs build on the earlier proofs, but are intuitive and can be grasped without the need of technical verifications.

2. The Menger game

A topological space has *Menger's property* $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ if, for each sequence of open covers, $\mathcal{U}_1,\mathcal{U}_2,\dots,$ we can select finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{F}_2,\dots$ whose union $\bigcup_n \mathcal{F}_n$ covers the space. The symbol O in this notation indicates that we are provided with open covers, and need to obtain an open cover. These properties were also considered for additional classes of open covers, in the realm of selection principles [13].

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Menger's game $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ is a game for two players, Alice and Bob, with an inning per each natural number n. In each inning, Alice picks an open cover of the ambient space and Bob selects finitely many members from this cover. Bob wins if the sets he selected throughout the game cover the space. If this is not the case, Alice wins.

If Alice does not have a winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, then $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ holds. The converse implication is a deep theorem of Hurewicz [5, Theorem 10]. This theorem is often used within selection principles, and has applications in diverse contexts, such as D-spaces [1] and additive Ramsey theory [12]. We present here a conceptual proof of this theorem. The proof uses the same initial simplifications as in Scheepers's proof of Hurewicz's Theorem [9, Theorem 13]. We add one further simplification that makes calculations easier, and an appropriate notion that goes through induction, and thus eliminates the necessity to track the history of the game.

Theorem 1 (Hurewicz)**.** *Let X be a space satisfying $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ space. Then Alice does not have a winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$.*

Proof. Fix an arbitrary strategy for Alice in the game $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$.

If there is a play in which Bob covers the space after finitely many steps, then we are done. Thus, we assume that, in no position, a finite selection suffices, together with the earlier selections, to cover the space. Since the space X satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, it is Lindelöf. By restricting Bob's moves to countable subcovers of Alice's covers, we may assume that Alice's covers are countable. Given that, we may assume that Alice's covers are *increasing*, that is, of the form $\{U_1,U_2,\dots\}$ with $U_1\subseteq U_2\subseteq\cdots$, and that Bob selects a *single* set in each move. Indeed, given a countable cover $\{U_1,U_2,\dots\}$, we can restrict Bob's selections to the form $\{U_1,U_2,\dots,U_n\}$, for $n\in\mathbb{N}$. Since Bob's goal is just to cover the space, we may pretend that Bob is provided covers of the form

$$\{U_1, U_1\cup U_2, U_1\cup U_2\cup U_3,\dots\},$$

and if Bob selects an element $U_1\cup\cdots\cup U_n$, he replies to Alice with the legal move $\{U_1,U_2,\dots,U_n\}$. If Bob manages to cover the space, this is not due to the unions.

Finally, we may assume that for each reply $\{U_1,U_2,\dots\}$ (with $U_1\subseteq U_2\subseteq\cdots$) of Alice's strategy to a move U, we have $U=U_1$. Indeed, we can transform the given cover into the cover $\{U, U\cup U_1, U\cup U_2,\dots\}$. If Bob chooses U, we provide Alice with the answer U_1, and if he chooses $U\cup U_n$, we provide Alice with the answer U_n. Since Bob has already chosen the set U, its addition in the new strategy does not help covering more points.

With these simplifications, Alice's strategy is identified with a tree of open sets, as follows: Alice's initial move is an open cover $\{U_1,U_2,\dots\}$. If Bob replies U_n, then Alice's move is $\{U_{n,1},U_{n,2},\dots\}$. In general, if Bob replies U_σ, for $\sigma\in\mathbb{N}^k$, then Alice's move is an increasing open cover

$$\mathcal{U}_\sigma := \{U_{\sigma(1),\dots,\sigma(k),1}, U_{\sigma(1),\dots,\sigma(k),2},\dots\},$$

with $U_\sigma = U_{\sigma(1),\dots,\sigma(k),1}$.

The proof will reduce to the following concept.

Definition 2. A countable cover $\mathcal{U}$ of a space X is a *tail cover* if the set of intersections of cofinite subsets of $\mathcal{U}$ is an open cover of X.

Equivalently, a cover $\{U_1, U_2, \dots\}$ is a tail cover if the family

$$\left\{ \bigcap_{n=1}^{\infty} U_n, \bigcap_{n=2}^{\infty} U_n, \dots \right\}$$

of intersections of cofinal segments of the cover is an open cover.

Lemma 3. *Let n be a natural number. Define $\mathcal{V}_n := \bigcup_{\sigma\in\mathbb{N}^n} \mathcal{U}_\sigma$. Then the family $\mathcal{V}_n$ is a tail cover of X.*

Proof. The proof is by induction on n.

The open cover $\mathcal{V}_1 = \mathcal{U}_{()}$ is increasing, and thus the set of cofinite intersections is again $\mathcal{V}_1$, an open cover of X.

Let n be a natural number. For brevity, enumerate $\mathcal{V}_n = \{V_1, V_2, \dots\}$, and

$$\mathcal{V}_{n+1} = \bigcup_{k=1}^{\infty} \{V_1^k, V_2^k, \dots\},$$

where

$$V_k = V_1^k \subseteq V_2^k \subseteq \cdots .$$

We assume, inductively, that the family $\mathcal{V}_n$ is a tail cover of X. Let $\mathcal{V}$ be a cofinite subset of $\mathcal{V}_{n+1}$. For each natural number k, let m_k be the minimal natural number with $V_{m_k}^k \in \mathcal{V}$. Then $\bigcap(\mathcal{V} \cap \{V_1^k, V_2^k, \dots\}) = V_{m_k}^k$ for all k, and $m_k = 1$ for all but finitely many natural numbers k. Let $I := \{\, k : m_k = 1 \,\}$, a cofinite subset of $\mathbb{N}$. We have

$$\bigcap \mathcal{V} = \bigcap_{k\in\mathbb{N}} (\mathcal{V} \cap \{V_1^k, V_2^k, \dots\}) = \bigcap_{k\in\mathbb{N}} V_{m_k}^k = \bigcap_{k\in I} V_k \cap \bigcap_{k\in\mathbb{N}\setminus I} V_{m_k}^k,$$

Since $\mathcal{V}_n$ is a tail cover, the set $\bigcap_{k\in I} V_k$ is open. The remaining part is a finite intersection of open sets. Thus, the set $\bigcap \mathcal{V}$ is open.

Let $x \in X$. Since $\mathcal{V}_n$ is a tail cover, the set $I := \{\, k : x \in V_k \,\}$ is cofinite. For $k \in \mathbb{N} \setminus I$, let m_k be the minimal natural number with $x \in V_{m_k}^k$. Then $x \in \bigcap_{k\in I} V_k \cap \bigcap_{k\in\mathbb{N}\setminus I} V_{m_k}^k$ and, as we saw, the latter set is an intersection of a cofinite subset of the family $\mathcal{V}_{n+1}$. □

For each n, let $\mathcal{V}_n'$ be the set of intersections of cofinite subsets of $\mathcal{V}_n$. Applying the property $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ to the sequence $\mathcal{V}_1', \mathcal{V}_2', \dots$, Bob obtains cofinite sets $\mathcal{W}_n \subseteq \mathcal{V}_n$ such that $X = \bigcup_n \bigcap \mathcal{W}_n$. In the n-th inning, Alice provides Bob with a cover that is an infinite subset of the family $\mathcal{V}_n$. Since the family $\mathcal{W}_n$ is cofinite in $\mathcal{V}_n$, Bob can choose an element $V_n \in \mathcal{V}_n \cap \mathcal{W}_n$. Then $X = \bigcup_n V_n$, and Bob wins.

This completes the proof of Hurewicz's Theorem. $\square$

To treat Rothberger's game, we need a result slightly stronger than Hurewicz's. The original proof of the following result, due to Pawlikowski [7, Lemma 1], is much more technical than the combination of the present proofs of Theorem 1 and Corollary 4.

We recall that Menger's property $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ is preserved by countable unions: Given a countable union of Menger spaces, and a sequence of open covers, we can split the sequence of covers into infinitely many disjoint subsequences, and use each subsequence to cover one of the given Menger spaces.

Corollary 4 (Pawlikowski)**.** *Let X be a space satisfying $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. For each strategy for Alice in the game $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, there is a play according to this strategy,*

$$(\mathcal{U}_1, \mathcal{F}_1, \mathcal{U}_2, \mathcal{F}_2, \dots),$$

such that for each point $x \in X$ we have $x \in \bigcup \mathcal{F}_n$ for infinitely many n.

Proof. We apply a reduction of Scheepers, originally used to prove the analogous theorem for the game considered in Section 3 [10, Theorem 3].

The product space $X \times \mathbb{N}$, a countable union of Menger spaces, satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. We define a strategy for Alice in the game $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, played on the space $X \times \mathbb{N}$. Let $\mathcal{U}$ be Alice's first move in the original game. Then, in the new game, her first move is

$$\tilde{\mathcal{U}} := \{\, U \times \{n\} : U \in \mathcal{U}, n \in \mathbb{N} \,\}.$$

If Bob selects a finite set $\tilde{\mathcal{F}} \subseteq \tilde{\mathcal{U}}$, we take the set

$$\mathcal{F} := \left\{\, U \in \mathcal{U} : \text{there is } n \text{ with } U \times \{n\} \in \tilde{\mathcal{F}} \,\right\}$$

as a move in the original game. Then Alice replies with a cover $\mathcal{V}$, and we continue in the same manner. By Hurewicz's Theorem, there is a play

$$(\tilde{\mathcal{U}}_1, \tilde{\mathcal{F}}_1, \tilde{\mathcal{U}}_2, \tilde{\mathcal{F}}_2, \dots)$$

in the new game, with $\bigcup_n \tilde{\mathcal{F}}_n$ a cover of $X \times \mathbb{N}$. Consider the corresponding play in the original strategy,

$$(\mathcal{U}_1, \mathcal{F}_1, \mathcal{U}_2, \mathcal{F}_2, \dots).$$

Let $x \in X$. There is a natural number n_1 with $(x,1) \in \bigcup \tilde{\mathcal{F}}_{n_1}$. Then $x \in \bigcup \mathcal{F}_{n_1}$. The set

$$F := \left\{\, k \in \mathbb{N} : \text{there is } U \text{ with } U \times \{k\} \in \bigcup_{i=1}^{n_1} \tilde{\mathcal{F}}_i \,\right\}$$

is finite. Let m be a natural number greater than all elements of the set F. There is a natural number n_2 with $(x,m) \in \bigcup \tilde{\mathcal{F}}_{n_2}$. Then $x \in \bigcup \mathcal{F}_{n_2}$, and $n_1 < n_2$. Continuing in a similar manner, we see that $x \in \bigcup \mathcal{F}_n$ for infinitely many n. $\square$

Remark 5. A cover of a space is *large* if each point is covered by infinitely many members of the cover. Let Λ be the family of all large covers of an ambient space X. We have $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O}) = \mathsf{S}_{\mathrm{fin}}(\Lambda,\Lambda)$ ([9, Corollary 5], [6, Theorem 1.2]). With some initial simplifications of the considered strategies, Corollary 4 implies that a topological space X satisfies $\mathsf{S}_{\mathrm{fin}}(\Lambda,\Lambda)$ if and only if Alice does not have a winning strategy in the corresponding game $\mathsf{G}_{\mathrm{fin}}(\Lambda,\Lambda)$. In fact, these results are essentially identical.

3. The Rothberger game

The definitions of *Rothberger's property* $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ and the corresponding game $\mathsf{G}_1(\mathrm{O},\mathrm{O})$ are similar to those of $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ and $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, respectively, but here we select *one* element from each cover. Here too, if Alice does not have a winning strategy then the space satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$. The converse implication was established by Pawlikowski [7, p. 279], improving considerably over partial results of Galvin [4, Corollary 4] and Recław [8, Corollary 2]. We provide a conceptual proof of Pawlikowski's Theorem. We first isolate an argument in Pawlikowski's proof, that does not involve games.

Lemma 6 (Pawlikowski)**.** *Let X be a space satisfying $\mathsf{S}_1(\mathrm{O},\mathrm{O})$. Let $\mathcal{F}_1,\mathcal{F}_2,\dots$ be nonempty finite families of open sets such that, for each point $x \in X$, we have $x \in \bigcup \mathcal{F}_n$ for infinitely many n. Then there are elements $U_1 \in \mathcal{F}_1, U_2 \in \mathcal{F}_2,\dots$ such that the family $\{U_1,U_2,\dots\}$ covers the space X.*

Proof. For each natural number n, let $\mathcal{U}_n$ be the family of all intersections of n open sets taken from distinct members of the sequence $\mathcal{F}_1,\mathcal{F}_2,\dots$. Then $\mathcal{U}_n$ is an open cover of X.

By the property $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, there are elements $V_1 \in \mathcal{U}_1, V_2 \in \mathcal{U}_2,\dots$ that cover the space X. Extend the set V_1 to an element of some family $\mathcal{F}_n$. We can extend V_2 to an element of some *other* family $\mathcal{F}_n$, and so on. We obtain a selection of at most element from each family $\mathcal{F}_n$, that covers X. We can extend our selection to have an element from each family $\mathcal{F}_n$. $\square$

For a natural number k and families of sets $\mathcal{U}_1,\dots,\mathcal{U}_k$, let

$$\mathcal{U}_1 \wedge \dots \wedge \mathcal{U}_k := \{\, U_1 \cap \dots \cap U_k : U_1 \in \mathcal{U}_1,\dots,U_k \in \mathcal{U}_k \,\}.$$

Theorem 7 (Pawlikowski)**.** *Let X be a space satisfying $\mathsf{S}_1(\mathrm{O},\mathrm{O})$. Then Alice does not have a winning strategy in the game $\mathsf{S}_1(\mathrm{O},\mathrm{O})$.*

Proof. Fix an arbitrary strategy for Alice in the Rothberger game $\mathsf{G}_1(\mathrm{O},\mathrm{O})$. Since $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ spaces are Lindelöf, we may assume that each cover in the strategy is countable. Let $\mathbb{N}^{<\infty}$

be the set of finite sequences of natural numbers. We index the open covers in the strategy as

$$\mathcal{U}_\sigma = \{U_{\sigma,1}, U_{\sigma,2}, \dots\},$$

for $\sigma \in \mathbb{N}^{<\infty}$, so that $\mathcal{U} = \{U_1, U_2, \dots\}$ is Alice's first move, and for each finite sequence $k_1, \dots, k_n$ of natural numbers, $\mathcal{U}_{k_1,\dots,k_n}$ is Alice's reply to the position

$$(\mathcal{U}, U_{k_1}, \mathcal{U}_{k_1}, U_{k_1,k_2}, \mathcal{U}_{k_1,k_2}, \dots, U_{k_1,\dots,k_n}).$$

For finite sequences $\tau, \sigma \in \mathbb{N}^n$, we write $\tau \leq \sigma$ if $\tau(i) \leq \sigma(i)$ for all $i = 1, \dots, n$. We define a strategy for Alice in the Menger game $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. Alice's first move is $\mathcal{U}$, her first move in the original strategy. Assume that Bob selects a finite subset $\mathcal{F}$ of $\mathcal{U}$. Let m_1 be the minimal natural number with $\mathcal{F} \subseteq \{U_1, \dots, U_{m_1}\}$. Then, in the Menger game, Alice's response is the joint refinement $\mathcal{U}_1 \wedge \dots \wedge \mathcal{U}_{m_1}$. Assume that Bob chooses a finite subset $\mathcal{F}$ of this refinement. Let m_2 be the minimal natural number such that $\mathcal{F}$ refines all sets $\{U_{i,1}, \dots, U_{i,m_2}\}$, for $i = 1, \dots, m_1$. Then Alice's reply is the joint refinement $\bigwedge_{\tau \leq (m_1,m_2)} \mathcal{U}_\tau$. In general, Alice provides a cover of the form $\bigwedge_{\tau \leq \sigma} \mathcal{U}_\tau$, for $\sigma \in \mathbb{N}^{<\infty}$, Bob selects a finite family refining all families $\{U_{\tau,1}, \dots, U_{\tau,m}\}$ for $\tau \leq \sigma$, whit the minimal natural number m, and Alice replies $\bigwedge_{\tau \leq (\sigma,m)} \mathcal{U}_\tau$.

By Pawlikowski's Theorem (Corollary 4), there is a play

$$(\mathcal{U}, \mathcal{F}_1, \bigwedge_{k_1 \leq m_1} \mathcal{U}_k, \mathcal{F}_2, \bigwedge_{(k_1,k_2) \leq (m_1,m_2)} \mathcal{U}_{k_1,k_2}, \dots),$$

according to the new strategy, such that every point of the space is covered infinitely often in the sequence $\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \dots$. By Lemma 6, we can pick one element from each set $\mathcal{F}_n$ and cover the space. There is $k_1 \leq m_1$ such that the first picked element is a subset of U_{k_1}. There is $k_2 \leq m_2$ such that the second picked element is a subset of U_{k_1,k_2}, and so on. Then the play

$$(\mathcal{U}, U_{k_1}, \mathcal{U}_{k_1}, U_{k_2,k_2}, \dots)$$

is in accordance with Alice's strategy in the Rothberger game, and is won by Bob. □

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