SUMMARY OF PROFESSIONAL ACCOMPLISHMENTS:

PIOTR SZEWCZAK

1. Diplomas and degrees

PhD. Mathematics 2015
Cardinal Stefan Wyszyński University, Poland
Thesis: *Generalized ordered spaces and paracompactness in Cartesian products*
Supervisor: dr hab. Kazimierz Alster

Msc. Mathematics 2008
Cardinal Stefan Wyszyński University, Poland
Thesis: *On the class of spaces whose Cartesian product with every paracompact space is paracompact*
Supervisor: dr hab. Kazimierz Alster

2. Information on employment in research institutes or faculties/departments

Assistant Professor .... 10.2016 – present
Cardinal Stefan Wyszyński University, Poland

Post-Doctoral Position .... 02.2016 – 01.2017
Bar-Ilan University, Israel

Teaching Assistant .... 10.2008 – 09.2016
Cardinal Stefan Wyszyński University, Poland

3. Description of the achievements: Combinatorial covering properties

3.0. Cycle of scientific articles related thematically.

(1) P. Szewczak, *Abstract colorings, games and ultrafilters*, Topology and its Applications 335 (2023) 108595, doi: 10.1016/j.topol.2023.108595, arXiv: 2107.02830.

(2) P. Szewczak, T. Weiss, *Null sets and combinatorial covering properties*, Journal of Symbolic Logic, 87 (2022), 1231 – 1242, doi:10.1017/jsl.2021.51, arXiv: 2006.10796.

(3) P. Szewczak, B. Tsaban, L. Zdomskyy, *Finite powers and products of Menger sets*, Fundamenta Mathematicae 253 (2021), 257 – 275, doi: 10.4064/fm896-4-2020, arXiv: 1903.03170.

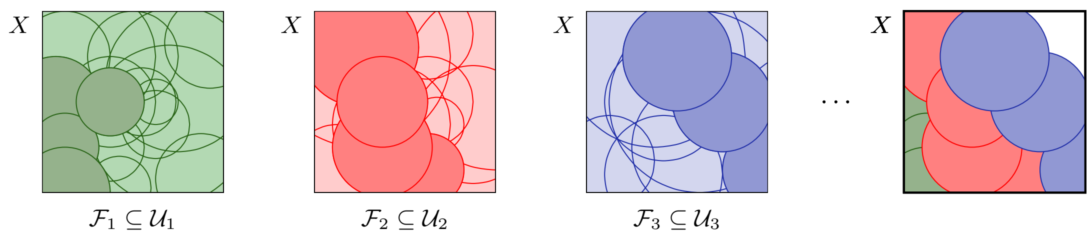
(4) P. Szewczak, M. Włudecka, *Unbounded towers and products*, Annals of Pure and Applied Logic 172 (2021), 102900, doi: 10.1016/j.apal.2020.102900, arXiv: 1912.02528.

(5) P. Szewczak, G. Wiśniewski, *Products of Luzin-type sets with combinatorial properties*, Topology and its Applications 264 (2019), 420–433, doi: 10.1016/j.topol. 2019.05.015, arXiv: 1903.05208.

(6) P. Szewczak, B. Tsaban, *Products of general Menger spaces*, Topology and its Applications 255 (2019), 41–55, doi: 10.1016/j.topol.2019.01.005, arXiv: 1607.01687.

(7) P. Szewczak, B. Tsaban, *Products of Menger spaces: a combinatorial approach*, Annals of Pure and Applied Logic 168 (2017), 1–18, doi: 10.1016/j.apal.2016.08.002, arXiv:1603.03361.

3.1. **Introduction.** In 1924, Menger [32] observed that any metric space X which is *σ-compact* (i.e., it is a countable union of its compact subsets) has such a property that for any basis $\mathcal{B}$ of X, there are sets $B_1, B_2, \ldots \in \mathcal{B}$, such that $\lim_{n\to\infty} \operatorname{diam}(B_n) = 0$ and $X = \bigcup_{n\in\mathbb{N}} B_n$. Menger conjectured [32] that the above property characterizes σ-compactness in the class of metric spaces. Hurewicz [26] reformulated the Menger property without using a metric: for any sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of a given topological space, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that the family $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an open cover of the space. In that way, the definition of the Menger property was extended on all topological spaces. Sierpiński [26] showed that a *Luzin set*, i.e., an uncountable subspace of the real line with the standard topology whose intersection with any meager set (Baire first category) is at most countable, is a counterexample to the Menger conjecture. The existence of a Luzin set is independent from ZFC and it can be constructed, e.g., assuming that the Continuum Hypothesis holds. In 2006, Bartoszyński and Tsaban [9], using some ideas from the work of Fremlin and Miller [21], presented a uniform construction in ZFC of a subspace of the real line which is Menger but no σ-compact.



The Menger property scheme

This is a brief description of the Menger property, one of the classic properties in the considered topic. *Combinatorial covering properties* are uniform procedures for generating a cover of a topological space from a sequence of covers [48]. They capture various properties defined in different mathematical fields like dimension theory, measure theory, and function spaces. This discipline connects topology, set theory, and functional analysis, and often makes it possible to transport methods between these fields. It is now one of the most active streams of research within the set-theoretic topology.

By *space* we mean an infinite Tikhonov topological space. A *cover* of a space is a family of sets whose union is the entire space. Let $\mathcal{U}$ be a cover of a space X. The cover $\mathcal{U}$ is a *γ-cover* if it is infinite and for any point $x \in X$ the set $\{\, U \in \mathcal{U} : x \notin U \,\}$ is finite. The cover $\mathcal{U}$ is an *ω-cover* if $X \notin \mathcal{U}$ and for any finite set $F \subseteq X$ there is a set $U \in \mathcal{U}$ with $F \subseteq U$. For a space, let O, Γ, Ω be families of all open covers, open γ-covers and open ω-covers of that space. For families $\mathcal{A}, \mathcal{B} \in \{\mathrm{O}, \Gamma, \Omega\}$ we define the following properties[1] a space might possess.

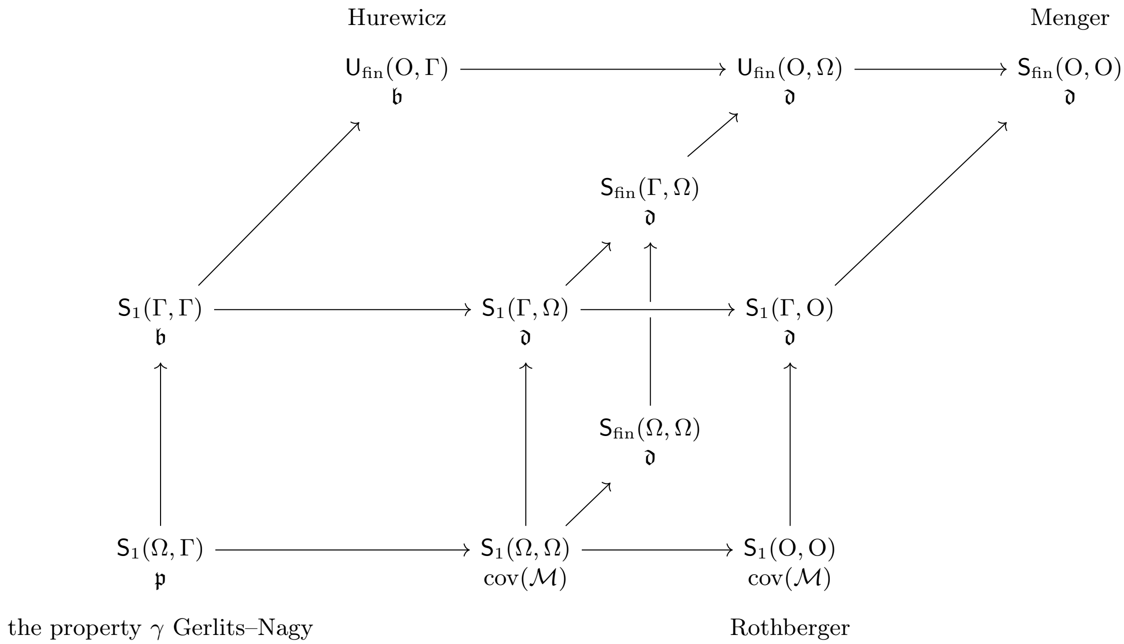
$\mathsf{S}_1(\mathcal{A},\mathcal{B})$: for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{A}$, there are $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \ldots$, such that $\{\, U_n : n \in \mathbb{N} \,\} \in \mathcal{B}$,

$\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$: for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{A}$, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$, such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \mathcal{B}$,

$\mathsf{U}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$: for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{A}$, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$, such that $\{\, \bigcup \mathcal{F}_n : n \in \mathbb{N} \,\} \in \mathcal{B}$.

According to the above notation a space is Menger if it satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. The considered properties in the above form were suggested by Scheepers who presented the relations among them in the form of a diagram, nowadays named after him [48]. The extreme properties in the diagram are classic and were introduced by Menger ($\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$) [32, 26], Hurewicz ($\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$) [26, 27], Rothberger ($\mathsf{S}_1(\mathrm{O},\mathrm{O})$) [43], Gerlits and Nagy ($\mathsf{S}_1(\Omega,\Gamma)$ (also known as the property γ) [23]. Some additional properties were studied by Arhangel'skiĭ ($\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$) [1], Sakai ($\mathsf{S}_1(\Omega,\Omega)$) [44], Bukovský ($\mathsf{S}_1(\Gamma,\Gamma)$) [14] and others, by relating them to previously well-studied local properties of topological function spaces.

[1]The properties $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ and $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ are defined also for arbitrary families $\mathcal{A}$ and $\mathcal{B}$.



The Scheepers diagram. Trivial properties or those which are equivalent in ZFC are not included.

The properties, considered here, are called *combinatorial*, since they have close connections with combinatorial structure of the *Baire* space $\mathbb{N}^{\mathbb{N}}$ [41] (we treat $\mathbb{N}$ as a discrete space and consider in $\mathbb{N}^{\mathbb{N}}$ the Tikhonov product topology). E.g., a zero-dimensional Lindelöf space is Menger if and only if no continuous image Y of the space into $\mathbb{N}^{\mathbb{N}}$ is dominating [27], i.e., there is a function $g \in \mathbb{N}^{\mathbb{N}}$, such that for any function $f \in Y$ the set $\{\, n : f(n) < g(n) \,\}$ is infinite. For each property $\mathbf{P}$ from the diagram we define a *critical cardinal number*, which is equal to the minimal cardinality of a subspace of $\mathbb{N}^{\mathbb{N}}$ which does not have the property $\mathbf{P}$. Since the space $\mathbb{N}^{\mathbb{N}}$ does not have the Menger property, which is the weakest among the above properties, the critical cardinal numbers are well defined for the considered properties and they are written in the diagram. The critical cardinal number for the Menger property is the dominating number $\mathfrak{d}$. It follows that any subspace of $\mathbb{N}^{\mathbb{N}}$ with the cardinality smaller than $\mathfrak{d}$ is Menger no matter what is the structure of this subspace. More details about cardinals of the continuum used in this presentation may be found in the survey of Blass [12]. The most common examples of spaces with combinatorial covering properties are *sets of reals*, i.e., spaces homeomorphic to subspaces of the real line with the standard topology. Combinatorics plays also a crucial role in constructing spaces with these properties.

Combinatorial covering properties found applications in such areas as forcing [18], function spaces [52], Ramsey theory in algebra [65], combinatorics of discrete subspaces [2], hyperspaces with the Vietoris topology [30], products of Lindelöf spaces [3] and products of paracompact spaces [56].

3.2. **Products of Menger sets** [57, 59]**.** Todorčević [68, §3] proved in ZFC that there are general topological spaces with the property γ, whose product is not Menger. It shows that none of the properties from the diagram is closed under finite products. In the class of sets of reals, the situation is much more subtle, and it deeply depends on set theory. By *set* with a topological property $\mathbf{P}$ we mean a set of reals (with the property $\mathbf{P}$). Assuming that the Continuum Hypothesis holds, Miller, Tsaban and Zdomskyy [36] constructed two sets with the property γ, whose product is not Menger. In contrast to this result, in the Laver model all sets of reals with properties from the lowest row, in the diagram, are countable [31]. Then, trivially, those properties are closed under finite products in the class of sets of reals

The problem, whether in ZFC there are Menger sets whose product is not Menger was suggested by Scheepers (*Open problems in Topology* [69, Problem 6.7]). Just, Miller, Scheepers and Szeptycki proved that if the Continuum Hypothesis holds, then such sets exist [28, Theorem 3.7]. Then Scheepers and Tsaban showed, using similar methods, that to this end the equality $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$ is sufficient [53, Theorem 49]. A $\mathrm{cov}(\mathcal{M})$-*Luzin set* is a space homeomorphic with a subspace X of the real line, such that $|X| \geq \mathrm{cov}(\mathcal{M})$ and $|X \cap M| < \mathrm{cov}(\mathcal{M})$ for all meager subsets M of the real line. The above examples are $\mathrm{cov}(\mathcal{M})$-Luzin sets (they are even hereditary

Rothberger; the existence of these sets is independent from ZFC). Another construction of Menger sets whose product is not Menger was provided by Repovš and Zdomskyy [42, Proposition 3.4] under $\mathfrak{b} = \mathfrak{d}$ (a detailed description of these sets will be presented later). During investigations, Zdomskyy [73] proved that in the Miller model the product of any two Menger sets is Menger. This results shows that in order to construct Menger sets with a non-Menger product, additional set-theoretic assumptions are necessarily.

We identify the Cantor cube $\{0,1\}^{\mathbb{N}}$ with the family $\mathrm{P}(\mathbb{N})$ of all subsets of the set of natural numbers $\mathbb{N}$. Since the Cantor cube is homeomorphic with the Cantor set, each subspace of $\mathrm{P}(\mathbb{N})$ can be viewed as a set of reals. Let $[\mathbb{N}]^{\infty}$ be the family of all infinite subsets of $\mathbb{N}$ and Fin be the family of all finite subsets of $\mathbb{N}$. We identify each set $a \in [\mathbb{N}]^{\infty}$ with its increasing enumeration, an element of the Baire space $\mathbb{N}^{\mathbb{N}}$. Then we have the following inclusion of sets $[\mathbb{N}]^{\infty} \subseteq \mathbb{N}^{\mathbb{N}}$. Moreover, the topology of the space $[\mathbb{N}]^{\infty}$ coincides with the subspace topology induced by $\mathbb{N}^{\mathbb{N}}$. Depending on the interpretation, points of the space $[\mathbb{N}]^{\infty}$ are referred to as sets or functions. Let κ be an infinite cardinal number. For functions $a, b \in [\mathbb{N}]^{\infty}$ we write $a \leq^* b$, if the set $\{\, n : b(n) < a(n) \,\}$ is finite. A set $X \subseteq [\mathbb{N}]^{\infty}$ is *κ-unbounded*, if $|X| \geq \kappa$ and for any function $b \in [\mathbb{N}]^{\infty}$, we have $|\{\, x \in X : x \leq^* b \,\}| < \kappa$. A space X is *$\mathfrak{d}$-concentrated*, if $|X| \geq \mathfrak{d}$ and there is a countable set $D \subseteq X$, such that $|X \setminus U| < \mathfrak{d}$ for any open set U in X containing D. There is in ZFC a $\mathfrak{d}$-unbounded set X and then the set $X \cup \mathrm{Fin}$ is $\mathfrak{d}$-concentrated. A counterexample in ZFC for the Menger conjecture given by Bartoszyński and Tsaban [9], has such a structure. Also note that any $\mathfrak{d}$-concentrated set is Menger [70, Corollary 1.14] but no σ-compact. The mentioned above examples of Menger sets, in the context of products, are $\mathfrak{d}$-concentrated.

We introduce a purely combinatorial approach to products of Menger sets. We obtain examples using hypotheses milder than earlier ones, as well as examples using hypotheses that are incompatible with the Continuum Hypothesis. The proof method is new. If $X \subseteq \mathrm{P}(\mathbb{N})$, then we treat X as a space with the subspace topology of $\mathrm{P}(\mathbb{N})$.

Theorem 3.1 ([57, Theorem 2.7])**.** *Let $\kappa \in \{\mathrm{cf}(\mathfrak{d}), \mathfrak{d}\}$ and $X \subseteq [\mathbb{N}]^{\infty}$ be a set containing a κ-unbounded set. Then there is a $\mathfrak{d}$-concentrated set $Y \subseteq [\mathbb{N}]^{\infty}$ (in particular it is Menger), such that the product $X \times Y$ is not Menger.*

Now we discuss some set-theoretic assumptions which provide the existence of a Menger set satisfying the assumptions from Theorem 3.1. Let $\mathfrak{d}$ be a singular number and $X \subseteq [\mathbb{N}]^{\infty}$ be a $\mathrm{cf}(\mathfrak{d})$-unbounded set with cardinality $\mathrm{cf}(\mathfrak{d})$ (such a set exists in ZFC [57, discussion before Lemma 2.6]). Then X is a *trivial* Menger set, i.e., its cardinality is smaller than the cardinal number $\mathfrak{d}$, the critical cardinal number for the Menger property. It is the first example in the literature of a trivial Menger set whose product with a Menger set is not Menger. Another assumption is the inequality $\mathfrak{d} \leq \mathfrak{r}$, which can be used to construct a Menger set in $[\mathbb{N}]^{\infty}$, which is $\mathfrak{d}$-unbounded [57, Theorem 3.3, Theorem 3.2]. Also note that if $\mathfrak{d}$ is singular, then the inequality $\mathfrak{d} \leq \mathfrak{r}$ holds [57, discussion in the proof of (2)⇒(1) in Theorem 3.3]. The inequality $\mathfrak{d} \leq \mathfrak{r}$ follows from each of the following assumptions: $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$ (used by Scheepers and Tsaban) or $\mathfrak{b} = \mathfrak{d}$ (used by Repovš and Zdomskyy).

Theorem 3.1 and its consequences lead to the question, whether the inequality $\mathfrak{d} \leq \mathfrak{r}$ is necessarily for the existence of two Menger sets whose product is not Menger. We extend methods used in the proof of Theorem 3.1 in another work [59, Theorem 2.12], in order to prove the following result, which provides the negative answer to this question. Let $\mathrm{add}(\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O}))$ be the minimal cardinality of a family of Menger subsets of $\mathrm{P}(\mathbb{N})$ whose union is not Menger.

Theorem 3.2 ([59, Theorem 2.12])**.** *Let $X \subseteq [\mathbb{N}]^{\infty}$ be a set containing an $\mathrm{add}(\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O}))$-unbounded set. Then there is a Menger set $Y \subseteq [\mathbb{N}]^{\infty}$, such that the product $X \times Y$ is not Menger.*

There is in ZFC an $\mathrm{add}(\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O}))$-unbounded set in $[\mathbb{N}]^{\infty}$ of cardinality $\mathrm{add}(\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O}))$ [59, Lemma 2.15]. It follows that if $\mathrm{add}(\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})) < \mathfrak{d}$, then the set X from Theorem 3.1 can be a trivial Menger set. The inequality $\mathrm{add}(\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})) < \mathfrak{d}$ holds, e.g., in the Blass–Shelah model ([59, Theorem 2.12], [13]) or if $\mathfrak{d}$ is singular [64, Corollary 2.3(3)]. Also note that in the Blass–Shelah model, the inequality $\mathfrak{d} > \mathfrak{r}$ holds.

Main achievements

- If $\mathfrak{d} \leq \mathfrak{r}$ or $\mathrm{add}(\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})) < \mathfrak{d}$, then there are Menger sets X and Y, whose product $X \times Y$ is not Menger.

3.3. **Products of Menger sets with strong properties** [59, 61]**.** Under some additional assumptions, the sets X and Y from the end of the previous subsection can have stronger properties as having all finite powers Menger or being hereditary Menger. Before presenting further results, we need the following notions.

An *ultrafilter on* $\mathbb{N}$ is a maximal with respect to inclusion a nonempty set $U \subseteq \mathrm{P}(\mathbb{N})$, such that for every $a, b \in U$ we have $a \cap b \in U$ and for every $a \in U$ and $b \in \mathrm{P}(\mathbb{N})$, if $a \subseteq b$, then $b \in U$. Let $U \subseteq [\mathbb{N}]^\infty$ be an ultrafilter. For functions $a, b \in [\mathbb{N}]^\infty$, we write $a \leq_U b$, if $\{ n : a(n) \leq b(n) \} \in U$. A set $A \subseteq [\mathbb{N}]^\infty$ is $\leq_U$*-bounded*, if there is a function $b \in [\mathbb{N}]^\infty$, such that $a \leq_U b$ for all functions $a \in A$. Let $\mathfrak{b}(U)$ be the minimal cardinality of a subset of $[\mathbb{N}]^\infty$, which is not $\leq_U$-bounded. We have the following inequalities $\mathfrak{b} \leq \mathfrak{b}(U) \leq \mathrm{cf}(\mathfrak{d})$. A set $X \subseteq [\mathbb{N}]^\infty$ is a U*-scale*, if $|X| \geq \mathfrak{b}(U)$ and for each function $b \in [\mathbb{N}]^\infty$ we have $|\{ x \in X : x \leq_U b \}| < \mathfrak{b}(U)$. In ZFC there is a U-scale X [71, Lemma 2.9] and then all finite powers of the set $X \cup \mathrm{Fin}$ are Menger [71, Theorem 4.5].

Repovš and Zdomskyy [42, Theorem 3.3], proved under $\mathfrak{b} = \mathfrak{d}$ that there are a U-scale X and a V-scale Y for some ultrafilters $U, V \subseteq [\mathbb{N}]^\infty$, such that the product $(X \cup \mathrm{Fin}) \times (Y \cup \mathrm{Fin})$ is not Menger (in particular, all finite powers of the sets $X \cup \mathrm{Fin}$ and $Y \cup \mathrm{Fin}$ are Menger). It has been shown that the same assertion holds under a weaker assumption but the proof is much more complex.

Theorem 3.3 ([59, Theorem 2.5(3)])**.** *Assume that* $\mathfrak{d} \leq \mathfrak{r}$ *and* $\mathfrak{d}$ *is singular. Then there are a* U*-scale* X *and a* V*-scale* Y *for some ultrafilters* $U, V \subseteq [\mathbb{N}]^\infty$*, such that the product* $(X \cup \mathrm{Fin}) \times (Y \cup \mathrm{Fin})$ *is not Menger.*

Theorem 3.3 has found applications in products of function spaces. Let Y be a space and for each element $y \in Y$ define $\Omega_y := \{ A \subseteq Y : y \in \overline{A} \}$. The space Y has *countable fan tightness* [1], if for each $y \in Y$, the space Y satisfies $\mathsf{S}_{\mathrm{fin}}(\Omega_y, \Omega_y)$, i.e., for every element $y \in Y$ and sets $A_1, A_2, \ldots \subseteq Y$ with $y \in \bigcap_{n \in \mathbb{N}} \overline{A_n}$, there are finite sets $F_1 \subseteq A_1$, $F_2 \subseteq A_2$, $\ldots$, such that $y \in \overline{\bigcup_{n \in \mathbb{N}} F_n}$. This property generalizes the first axiom of countability. For a space X, let $\mathrm{C_p}(X)$ be a space of all continuous real-valued functions on X with the pointwise convergence topology. If X is a separable metric space, then the space $\mathrm{C_p}(X)$ has countable fan tightness if and only if X satisfies $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$ [52, Theorem 35] (equivalently, all finite powers of X are Menger [28, Theorem 3.9]). Then if X and Y are sets from Theorem 3.3, then the spaces $\mathrm{C_p}(X \cup \mathrm{Fin})$ and $\mathrm{C_p}(Y \cup \mathrm{Fin})$ have countable fan tightness but the product $\mathrm{C_p}(X \cup \mathrm{Fin}) \times \mathrm{C_p}(Y \cup \mathrm{Fin})$ does not have this property [59, Proposition 3.1(1)].

Assuming $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, many authors independently proved that there are two $\mathrm{cov}(\mathcal{M})$-Luzin sets, whose all finite powers are Rothberger and whose product is not Menger ([64, Proposition 3.1], [28, page 205], [51, Theorem 13], [29, Chapter 3], [8, Theorem 4]). Category theoretic methods used there, have substantial limitations, which prevent weakening assumptions. A natural question arises, whether the existence of such $\mathrm{cov}(\mathcal{M})$-Luzin sets can be proven under $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$ (which enables for a construction of a $\mathrm{cov}(\mathcal{M})$-Luzin set). It turns out that the answer is positive but with the additional assumption that the cardinal number $\mathrm{cov}(\mathcal{M})$ is regular. In particular, the following result shows that there are such sets in the Sacks model [47, 10], where $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \aleph_1 < \mathfrak{c}$. In this model a category theoretic approach fails.

Theorem 3.4 ([61, Theorem 2.1])**.** *Assume that* $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$ *and the cardinal number* $\mathrm{cov}(\mathcal{M})$ *is regular. Then there are a* U*-scale* X *and a* V*-scale* Y *for some ultrafilters* $U, V \subseteq [\mathbb{N}]^\infty$*, such that* $X \cup \mathrm{Fin}$ *and* $Y \cup \mathrm{Fin}$ *are* $\mathrm{cov}(\mathcal{M})$*-Luzin sets and the product* $(X \cup \mathrm{Fin}) \times (Y \cup \mathrm{Fin})$ *is not Menger*

For ultrafilter U and V from Theorem 3.4, we have $\mathfrak{b}(U) = \mathfrak{b}(V) = \mathrm{cov}(\mathcal{M})$, and thus all finite powers of the sets $X \cup \mathrm{Fin}$ and $Y \cup \mathrm{Fin}$ from Theorem 3.4 are Rothberger [59, Lemma 2.21]. This fact and Theorem 3.4 have found applications in products of function spaces [61, Corollary 4.1].

Main achievements

- If $\mathfrak{d} \leq \mathfrak{r}$ and the cardinal number $\mathfrak{d}$ is regular, then there are two sets whose all finite powers are Menger and whose product is not Menger.
- If $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$ and the cardinal number $\mathrm{cov}(\mathcal{M})$ is regular, then there are two $\mathrm{cov}(\mathcal{M})$-Luzin sets, whose all finite powers are Rothberger and whose product is not Menger.

3.4. **The Menger property parameterized by semifilters** [57]**.** A *semifilter* [4] is a set $S \subseteq [\mathbb{N}]^\infty$, such that for every sets $s \in S$ and $b \in [\mathbb{N}]^\infty$, if the set $b \setminus s$ is finite, then $b \in S$. Important examples of semifilters include the maximal semifilter $[\mathbb{N}]^\infty$, the minimal semifilter cF of all cofinite sets, and every ultrafilter $U \subseteq [\mathbb{N}]^\infty$ on $\mathbb{N}$. Let S be a semifilter. For functions $a, b \in [\mathbb{N}]^\infty$, we write $a \leq_S b$, if $\{ n : a(n) \leq b(n) \} \in S$. A set $A \subseteq [\mathbb{N}]^\infty$ is $\leq_S$*-bounded*, if there is a function $b \in [\mathbb{N}]^\infty$, such that $a \leq_S b$ for all functions $a \in A$. According to this notation, the relation $\leq^*$ on $[\mathbb{N}]^\infty$ is the same as $\leq_{\mathrm{cF}}$. Let $\mathfrak{b}(S)$ be the minimal cardinality of a subset of $[\mathbb{N}]^\infty$, which is not $\leq_S$-bounded. We have $\mathfrak{b}(\mathrm{cF}) = \mathfrak{b}$ and $\mathfrak{b}([\mathbb{N}]^\infty) = \mathfrak{d}$.

Let S be a semifilter. A space X is *S-Menger*, if for every sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of X, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{U}_2$, ..., such that the sets $\{\, n : x \in \bigcup \mathcal{F}_n \,\} \in S$ for all $x \in X$. In that terminology $[\mathbb{N}]^\infty$-Menger is Menger and cF-Menger is Hurewicz. We have the following implications:

$$\text{Hurewicz} \longrightarrow S\text{-Menger} \longrightarrow \text{Menger}.$$

The S-Menger property has the following combinatorial characterization. For a space X, a function $\Psi\colon X \to [\mathbb{N}]^\infty$ is *upper continuous*, if the sets $\{\, x \in X : \Psi(x)(n) \leq m \,\}$ are open for all natural numbers n and m. In the class of Lindelöf spaces, a space X is S-Menger if and only if every upper continuous image of X into $[\mathbb{N}]^\infty$ is $\leq_S$-bounded. [35, Theorem 7.3]. If we restrict our consideration to Lindelöf zero-dimensional spaces, then *upper continuous* can be replaced by *continuous*. In general, however, this is not the case. The properties considered here are hereditary for closed subsets. The planar set

$$X := ((\mathbb{R} \setminus \mathbb{Q}) \times [0,1]) \cup (\mathbb{R} \times \{1\}) \subseteq \mathbb{R}^2.$$

is not Menger, since the set $(\mathbb{R} \setminus \mathbb{Q}) \times \{0\}$ (homeomorphic with $[\mathbb{N}]^\infty$) is a closed subset of X and it is not Menger. Since the set X is connected, every continuous image of X into $[\mathbb{N}]^\infty$ is a singleton.

The forthcoming Lemma 3.5 is a main tool in proving many results about products of sets with the Menger property parameterized by semifilters, in the class of hereditary Lindelöf spaces [57, Sections 5 and 6] and in the class of all spaces [58]. Lemma 3.5 is a generalization of an earlier result of Miller, Tsaban and Zdomskyy [35, Lemma 6.3] to general topological spaces. The earlier proof [35, Lemma 6.3] does not apply in this general setting, and thus an alternative proof has been provided.

Lemat 3.5 ([57, Lemma 5.1])**.** *Let $X \subseteq [\mathbb{N}]^\infty$, Y be a space and $\Psi\colon (X \cup \mathrm{Fin}) \times Y \to [\mathbb{N}]^\infty$ be un upper continuous function. Then there is an upper continuous function $\Phi\colon Y \to [\mathbb{N}]^\infty$, such that for every points $x \in X$ and $y \in Y$ and a natural number n:*

$$\text{if } \Phi(y)(n) \leq x(n), \text{ then } \Psi(x,y)(n) \leq \Phi(y)(n).$$

For the sake of clarity of the presented results, we restrict our consideration to show some applications of Lemma 3.5 for the U-Menger property, where $U \subseteq [\mathbb{N}]^\infty$ is an ultrafilter, in the class of hereditary Lindelöf spaces. Tsaban and Zdomskyy showed, that for some special U-scales X, where $U \subseteq [\mathbb{N}]^\infty$ is an ultrafilter, all finite powers of the set $X \cup \mathrm{Fin}$ are U-Menger [71, Theorem 4.5]. This result follows from the following, much more general Theorem 3.6.

Theorem 3.6 ([57, Theorem 5.3(2)])**.** *Let $U \subseteq [\mathbb{N}]^\infty$ be an ultrafilter and X be a U-scale. In the class of hereditary Lindelöf space, the set $X \cup \mathrm{Fin}$ is productively U-Menger.*

If $\mathfrak{u} < \mathfrak{g}$, then for each ultrafilter $U \subseteq [\mathbb{N}]^\infty$, the properties U-Menger and Menger are equivalent in the class of sets of reals [66, dowód Theorem 3.7]. In contrast to this fact, we have the following result.

Theorem 3.7 ([57, Theorem 6.9])**.** *Assume that $\mathfrak{b} = \mathfrak{d}$ and U is an ultrafilter. In the class of hereditary Lindelöf spaces, there is a productively U-Menger set, which is not Hurewicza and not productively Menger. Moreover the U-Menger property is strictly between Hurewicz and Menger.*

Main achievements

- Lemma about products and upper continuous functions.
- For every ultrafilter $U \subseteq [\mathbb{N}]^\infty$ and U-scale X, the set $X \cup \mathrm{Fin}$ is productively U-Menger, in the class of hereditary Lindelöf spaces.
- If $\mathfrak{b} = \mathfrak{d}$, then for every ultrafilter $U \subseteq [\mathbb{N}]^\infty$, the U-Menger property is strictly between Hurewicz and Menger, in the realm of sets of reals.

3.5. **Separation of the Menger and Scheepers properties** [59]**.** In the combinatorial covering properties theory one of the major problems is to settle some additional relations between properties in the diagram in the class of sets of reals, using some set-theoretic assumptions. E.g., under $\mathfrak{u} < \mathfrak{g}$, the properties Menger and Scheepers $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Omega)$ are equivalent for sets of reals [66]. On the other hand, assuming $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$, there is a $\mathrm{cov}(\mathcal{M})$-Luzin set (in particular it is Menger), which is not Scheepers [53, Theorem 32]. Combinatorial methods used in proofs of the already presented results has been used also here.

Main achievements

- If $\mathfrak{d} \leq \mathfrak{r}$, then there is a Menger set which is not Scheepers.

3.6. **Productivity** [57, 58]. Let $\mathbf{P}$ be a topological property. A space X is *productively* $\mathbf{P}$ *in a given class of spaces*, if for every space Y form the class with the property $\mathbf{P}$, the product $X \times Y$ has the property $\mathbf{P}$. A space is *productively* $\mathbf{P}$, if it is productively $\mathbf{P}$ in the class of all spaces.

Miller, Tsaban and Zdomskyy proved, assuming $\mathfrak{d} = \aleph_1$, that in the class of hereditary Lindelöf spaces each productively Lindelöf set is productively Menger and productively Hurewicz [35, Theorem 8.2]. The below result establishes a relation between productively Menger and productively Hurewicz sets, in the class of hereditary Lindelöf spaces under $\mathfrak{b} = \mathfrak{d}$. The proof of this result applies Theorem 3.1.

Theorem 3.8 ([57, Theorem 4.8(2)]). *Assume that $\mathfrak{b} = \mathfrak{d}$. In the class of hereditary Lindelöf spaces, each productively Menger set is productively Hurewicz.*

The investigations were also related to productivity of properties in the class of all spaces. Under $\mathfrak{d} = \aleph_1$, Aurichi and Tall [3], improved earlier results by proving that each productively Lindelöf space is Hurewicz. The below Theorem 3.9 generalizes the result of Aurichi and Tall.

Theorem 3.9 ([58, Theorem 3.14]). *Assume that $\mathfrak{d} = \aleph_1$. Then each productively Lindelöf space is productively Menger and each productively Menger space is productively Hurewicz.*

In the light of Theorem 3.9, a natural question arises, whether the classes of productively Lindelöf, productively Menger and productively Hurewicz spaces are different, under $\mathfrak{d} = \aleph_1$. Let $X = \{\, x_\alpha : \alpha < \mathfrak{d} \,\} \subseteq [\mathbb{N}]^\infty$ be a *dominating scale* in $[\mathbb{N}]^\infty$, i.e., for any function $x \in [\mathbb{N}]^\infty$, there is an ordinal number $\alpha < \mathfrak{d}$, such that $x \leq^* x_\alpha$ and $x_\alpha \leq^* x_\beta$ for all ordinal numbers $\alpha < \beta < \mathfrak{d}$ (such a set exists, if $\mathfrak{b} = \mathfrak{d}$ [70, Lemma 1.4]). Miller, Tsaban and Zdomskyy proved that, in the class of hereditary Lindelöf spaces, the set $X \cup \mathrm{Fin}$ is productively Hurewicz and productively Menger [35, Theorem 6.5(1), Theorem 6.2]. A projection method, used in one of these results [35, Theorem 6.2] was applied together with so called Dedekind's compactifications of some spaces, in order to prove the following result. It is a new approach in the considered topic.

Theorem 3.10 ([58, Lemma 3.4, Corollary 3.6]). *Let X be a dominating scale in $[\mathbb{N}]^\infty$ of cardinality $\aleph_1$ and let $(X \cup \mathrm{Fin})_M$ be a set $X \cup \mathrm{Fin}$ with the following topology: the points from X are isolated and open neighborhoods of points from* Fin *are the same as in the natural topology induced by* $\mathrm{P}(\mathbb{N})$. *Then the space $(X \cup \mathrm{Fin})_M$ is productively Menger but not productively Lindelöf.*

Also note that it is consistent with $\mathfrak{d} = \aleph_1$ that there is a productively Menger space which is not productively Hurewicz [58, Proposition 3.15(2)].

Main achievements

- If $\mathfrak{b} = \mathfrak{d}$, then in the class of hereditary Lindelöf spaces, each productively Menger space is productively Hurewicz.
- Assume that $\mathfrak{d} = \aleph_1$. Then each productively Lindelöf space is productively Menger and each productively Menger space is productively Hurewicz. None of these implications is reversible.

3.7. **Unbounded towers** [62]. We already noticed that, if X is a $\mathfrak{d}$-unbounded set, then the set $X \cup \mathrm{Fin}$ is Menger. Bartoszyński and Shelah [7] showed that if $X = \{\, x_\alpha : \alpha < \mathfrak{b} \,\} \subseteq [\mathbb{N}]^\infty$ is a set which is not $\leq^*$-bounded and $x_\alpha \leq^* x_\beta$ for all ordinal numbers $\alpha < \beta < \mathfrak{b}$, then the set $X \cup \mathrm{Fin}$ is Hurewicz. It is the first uniform construction in ZFC of a Hurewicz set which is not σ-compact, a counterexample for the Hurewicz conjecture [26] that in the class of metric spaces the properties σ-compactness and Hurewicz are equivalent. The sets from the beginning of this paragraph are examples of nontrivial Menger and Hurewicz sets, respectively. It is not always the case that a construction of such nontrivial sets, for a given property, is possible. In the Laver model, each set with the property $\mathsf{S}_1(\Gamma, \Gamma)$ has cardinality smaller than $\mathfrak{b}$ [34, Corollary 4.4], and thus each such a set is trivial with respect to the property $\mathsf{S}_1(\Gamma,\Gamma)$. In that model, a similar situation holds for the property γ, which characterizes countable sets of reals [31].

Sometimes a combinatorial structure of sets with a given property **P** from the diagram does not provide that their product has also the property **P**. Results from Subsection 3.2 show that, under some assumptions, there are $\mathfrak{d}$-unbounded sets X and Y, such that the product $(X \cup \mathrm{Fin}) \times (Y \cup \mathrm{Fin})$ is not Menger. On the other hand the set $X \cup \mathrm{Fin}$, from the result of Bartoszyński and Shelah, is even productively Hurewicz, in the class of hereditary Lindelöf spaces [57, Theorem 5.4]. In the remaining part of this subsection, we present results related to products of nontrivial sets with the properties $\mathsf{S}_1(\Gamma,\Gamma)$ or γ, having some specific combinatorial structure.

Let κ be an infinite cardinal number. A set $\{\, x_\alpha : \alpha < \kappa \,\} \subseteq [\mathbb{N}]^\infty$ is a *κ-unbounded tower* [34], if it is not $\leq^*$-bounded and the sets $x_\alpha \setminus x_\beta$ are finite for all ordinal numbers $\beta < \alpha < \kappa$. Let X be a $\mathfrak{b}$-unbounded tower (such a set exists, e.g., assuming $\mathfrak{p} = \mathfrak{b}$ or $\mathfrak{b} < \mathfrak{d}$ [34, Lemma 2.2]). Then the set $X \cup \mathrm{Fin}$ has the property $\mathsf{S}_1(\Gamma,\Gamma)$ [34, Proposition 2.5]. Let $\mathrm{add}(\mathsf{S}_1(\Gamma,\Gamma))$ be the minimal cardinality of a family of subsets of $\mathrm{P}(\mathbb{N})$ with the property $\mathsf{S}_1(\Gamma,\Gamma)$, whose union is not $\mathsf{S}_1(\Gamma,\Gamma)$. Miller and Tsaban proved, assuming $\mathrm{add}(\mathsf{S}_1(\Gamma,\Gamma)) = \mathfrak{b}$, that all finite powers of the set $X \cup \mathrm{Fin}$ satisfy $\mathsf{S}_1(\Gamma,\Gamma)$ [34, Theorem 2.8]. Let Γ_{Bor} be a family of all countable γ-covers consisting of Borel sets of a given space. Miller, Tsaban and Zdomskyy proved that if X is a $\mathfrak{b}$-unbounded tower and Y is a set with the property $\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})$, then the product $(X \cup \mathrm{Fin}) \times Y$ is $\mathsf{S}_1(\Gamma,\Gamma)$ [36, Theorem 7.1]. An example of a set with the property $\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})$ is a *Sierpiński set* ([28, Theorem 2.9], [70, Theorem 2.4]), i.e., an uncountable subset of the real line whose intersection with any Lebesgue measure zero set is at most countable. All the above results have been generalized.

Theorem 3.11 ([62, Theorem 3.1])**.** *Let n be a natural number, $X_1, \dots, X_n$ be $\mathfrak{b}$-unbounded towers and Y be a set with the property $\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})$. Then the product*

$$(X_1 \cup \mathrm{Fin}) \times \cdots \times (X_n \cup \mathrm{Fin}) \times Y$$

has the property $\mathsf{S}_1(\Gamma,\Gamma)$.

The properties $\mathsf{S}_1(\Gamma,\Gamma)$ and $\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})$ are closely related to local properties of functions spaces. Let X be a space. A sequence $f_1, f_2, \dots \in \mathrm{C_p}(X)$ converges *quasinormally* to the constant zero function $\mathbf{0}$, if there is a sequence of positive real numbers $\epsilon_1, \epsilon_2, \dots$ converging to zero such that for any point $x \in X$, we have $|f_n(x)| < \epsilon_n$ for all but finitely many natural numbers n. A space X is a *QN-space* (*wQN-space*) if every sequence $f_1, f_2, \dots \in \mathrm{C_p}(X)$ converging pointwise to $\mathbf{0}$, converges (has a subsequence converging) quasinormally to $\mathbf{0}$. By a breakthrough result of Tsaban and Zdomskyy [67, Theorem 2], a subset of $\mathrm{P}(\mathbb{N})$ is a QN-space if and only if it satisfies $\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})$ [67, Corollary 20]. Every perfectly normal space satisfying $\mathsf{S}_1(\Gamma,\Gamma)$, is a wQN-space [16, Theorem 7]. QN-spaces, wQN-spaces and their variations were extensively studied by Bukovský, Haleš, Recław, Sakai and Scheepers [15, 16, 17, 24, 45, 46, 50].

By the result of Tsaban and Orensthein, if X is a $\mathfrak{p}$-unbounded tower (which existence is equivalent to $\mathfrak{p} = \mathfrak{b}$ [39, Lemma 3.3]), then the set $X \cup \mathrm{Fin}$ has the property γ ([39, Theorem 3.6], [40, Theorem 6]). Miller, Tsaban and Zdomskyy proved that, if X is an $\aleph_1$-unbounded tower, then the set $X \cup \mathrm{Fin}$ is productively γ in the class of sets of reals [36, Theorem 2.8]. These results have been generalized to the following form.

Theorem 3.12 ([62, Theorem 4.1(1)])**.** *If n is a natural number, $X_1, \dots, X_n$ are $\mathfrak{p}$-unbounded towers, then the product*

$$(X_1 \cup \mathrm{Fin}) \times \cdots \times (X_n \cup \mathrm{Fin})$$

has the property γ.

Theorem 3.13 ([62, Theorem 4.1(2)])**.** *If X is a $\mathfrak{p}$-unbounded tower and each subset of $\mathrm{P}(\mathbb{N})$ of cardinality smaller than $\mathfrak{p}$ is productively γ, in the class of sets of reals, then the set $X \cup \mathrm{Fin}$ is productively γ, in the class of sets of reals.*

Main achievements

- If n is a natural number, $X_1, \dots, X_n$ are $\mathfrak{b}$-unbounded towers and Y is a set with the property $\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})$, then the product

$$(X_1 \cup \mathrm{Fin}) \times \cdots \times (X_n \cup \mathrm{Fin}) \times Y$$

has the property $\mathsf{S}_1(\Gamma,\Gamma)$.

- If n is a natural number, $X_1, \dots, X_n$ are $\mathfrak{p}$-unbounded towers, then the product
$$(X_1 \cup \mathrm{Fin}) \times \dots \times (X_n \cup \mathrm{Fin})$$
has the property γ.
- If X is a $\mathfrak{p}$-unbounded tower and each subset of $\mathrm{P}(\mathbb{N})$ of cardinality smaller than $\mathfrak{p}$ is productively γ, in the class of sets of reals, then the set $X \cup \mathrm{Fin}$ is productively γ, in the class of sets of reals.

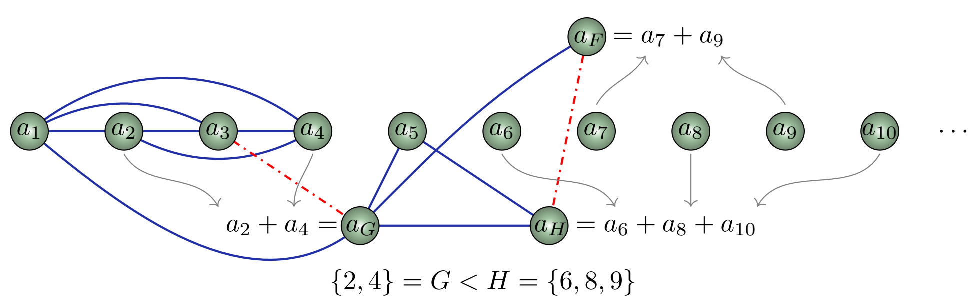
3.8. **Zero-additive sets** [60]**.** The space $\mathrm{P}(\mathbb{N})$ with the symmetric difference operation $\oplus$ is a topological group. Consider in $\mathrm{P}(\mathbb{N})$ the productive Lebesgue measure. A set $X \subseteq \mathrm{P}(\mathbb{N})$ is *zero-additive*, if for each measure zero set $N \subseteq \mathrm{P}(\mathbb{N})$, the set $X \oplus N := \{\, x \oplus y : x \in X, y \in N \,\}$ is null. Galvin and Miller, assuming that the Martin Axiom holds, proved that there is a subset of $\mathrm{P}(\mathbb{N})$ of cardinality $\mathfrak{p}$ with the property γ and each continuous image of that set into $\mathrm{P}(\mathbb{N})$ is zero-additive [22, Theorem 7]. Bartoszyński and Recław [6], under $\mathfrak{p} = \mathfrak{c}$, constructed a subset of $\mathrm{P}(\mathbb{N})$ of cardinality $\mathfrak{p}$ with the property γ, which is not zero-additive. Ideas from the previous subsection were combined with a characterization of zero-additive sets given by Shelah [54] and results of Zindulka [74] in order to weaken the assumptions in the above mentioned results. The proof method used here is different than in previous results.

Let $\mathrm{non}(\mathcal{N}_{\mathrm{add}})$ be the minimal cardinality of a subsets of $\mathrm{P}(\mathbb{N})$, which is not zero-additive.

Main achievements

- If $\mathfrak{p} = \mathrm{non}(\mathcal{N}_{\mathrm{add}}) = \mathfrak{c}$, then there is a subset of $\mathrm{P}(\mathbb{N})$ cardinality $\mathfrak{p}$ with the property γ, whose all continuous images into $\mathrm{P}(\mathbb{N})$ are zero-additive, and it contains a homeomorphic copy a non zero-additive set.
- If $\mathfrak{p} = \mathfrak{b} = \mathrm{non}(\mathcal{N})$, then there is a subset of $\mathrm{P}(\mathbb{N})$ of cardinality $\mathfrak{p}$ with the property γ, which is not zero-additive.

3.9. **Theorems about colorings** [55]**.** A *coloring* of a nonempty set X is a function $\chi\colon X \to \{1, \dots, k\}$, for some natural number k. For a coloring χ of a nonempty set X, a set $A \subseteq X$ is χ*-monochromatic*, if there is a natural number i with $\chi[A] = \{i\}$ (if the coloring χ is clear from the context, then the set A is just *monochromatic*). By the celebrated Hindman Finite Sums Theorem [25], for each coloring of $\mathbb{N}$, there is an infinite set $A \subseteq \mathbb{N}$, such that all finite sums of pairwise different elements from A have the same color. There are also generalizations of this Theorem to higher dimensions. By $[\mathbb{N}]^2$ we denote the set of all two-element subsets of $\mathbb{N}$, equivalently, the edge set of the complete graph with vertices in $\mathbb{N}$. Let $\mathrm{Fin}(\mathbb{N})$ be the set of all nonempty finite subsets of $\mathbb{N}$. Let $+$ be the usual addition in $\mathbb{N}$ and take a sequence $a_1, a_2, \dots \in \mathbb{N}$. For a set $G = \{i_1, \dots, i_n\} \in \mathrm{Fin}(\mathbb{N})$, where n is a natural number and $i_1 < \dots < i_n$, define $a_G := a_{i_1} + \dots + a_{i_n}$ For sets $G, H \in \mathrm{Fin}(\mathbb{N})$, we write $G < H$, if $\max G < \min F$. The sequence $a_1, a_2, \dots$ is *proper*, if for every sets $G, H \in \mathrm{Fin}(\mathbb{N})$ with $G < H$, we have $a_H \neq a_G$. A *sumgraph* of the proper sequence $a_1, a_2, \dots \in \mathbb{N}$ is the set
$$\{\, \{a_G, a_H\} : G, H \in \mathrm{Fin}(\mathbb{N}), G < H \,\}.$$



The scheme of the sumgraph of a proper sequence $a_1, a_2, \dots$

The Milliken–Taylor Theorem [37, 63] is a generalization of the Hindman Theorem: for each coloring of $[\mathbb{N}]^2$, there is a proper sequence in $\mathbb{N}$ with a monochromatic sumgraph.

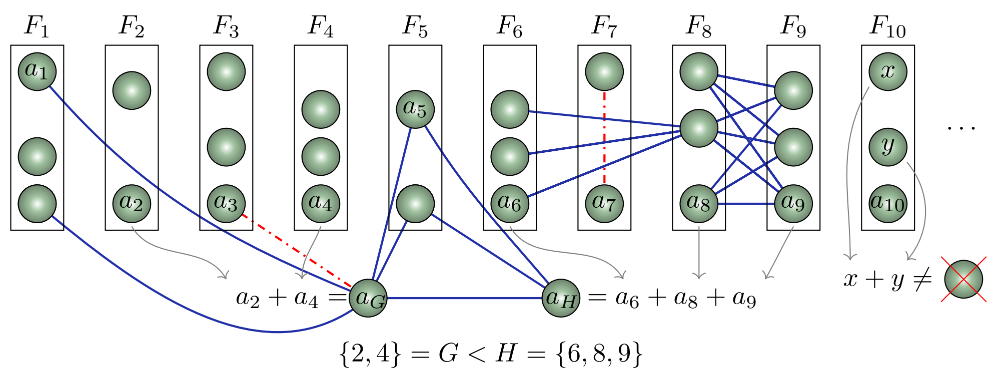
In proofs of results in a similar spirit we use the *Čech–Stone compactification* $\beta\mathbb{N}$ of a space $\mathbb{N}$ with the discrete topology, i.e., the set of all ultrafilters on $\mathbb{N}$ with a topology generated by the sets $\{\, p \in \beta\mathbb{N} : A \in p \,\}$, where $A \subseteq \mathbb{N}$. The usual addition $+$ on $\mathbb{N}$ can be extended to an operation on $\beta\mathbb{N}$, also denoted by $+$, such that for each $q \in \beta\mathbb{N}$ the map $p \mapsto p + q$ is continuous. The set $\mathbb{N}$ with the operation $+$ is a *semigroup*, i.e., the operation is associative. Also $\beta\mathbb{N}$ with the extended operation $+$ is a semigroup. By the Ellis–Numakura Lemma [20, 38], each nonempty compact subsemigroup of $\beta\mathbb{N}$ contains an *idempotent*, i.e., there is in this semigroup an ultrafilter e with $e + e = e$. Proofs of the mentioned in this subsection results are based on the existence of an idempotent in $\beta\mathbb{N}$.

An ultrafilter $p \in \beta\mathbb{N}$ is *large* for a nonempty family $\mathcal{R} \subseteq \mathrm{Fin}(\mathbb{N})$, if for each set $A \in p$, there is a set $R \in \mathcal{R}$ with $R \subseteq A$. An ultrafilter $p \in \beta\mathbb{N}$ is *large for a sequence* $\mathcal{R}_1, \mathcal{R}_2, \ldots \subseteq \mathrm{Fin}(\mathbb{N})$ of nonempty families, if it is large for each family from the sequence. E.g., in $\beta\mathbb{N}$ there is an idempotent large for the sequence $\mathcal{R}_1, \mathcal{R}_2, \ldots \subseteq \mathrm{Fin}(\mathbb{N})$ of families of increasing arithmetic progressions (we identify a sequence with its image) of length $1, 2, \ldots$, respectively [19, Lemma 2]. A *partite graph* of a sequence $F_1, F_2, \ldots \in \mathrm{Fin}(\mathbb{N})$ of pairwise disjoint sets is the set

$$\left\{\, \{a, b\} : a \in F_i, b \in F_j, i \neq j, i, j \in \mathbb{N} \,\right\}.$$

Let $F_1, F_2, \ldots \in \mathrm{Fin}(\mathbb{N})$ be a sequence of sets such that all sequences in the product $F_1 \times F_2 \times \cdots$ are proper. A *partite sumgraph of the sequence* $F_1, F_2, \ldots \in \mathrm{Fin}(\mathbb{N})$ is the set

$$\left\{ \{a_G, a_H\} : a_1, a_2, \ldots \in F_1 \times F_2 \times \cdots, G, H \in \mathrm{Fin}(\mathbb{N}), G < H \right\}$$



The scheme of the partite sumgraph of a sequence $F_1, F_2, \ldots$

The following Theorem due to Bergelson and Hindman [11] shows that a structure of monochromatic sets for colorings of $[\mathbb{N}]^2$ can be very rich. For each coloring of $[\mathbb{N}]^2$, there are increasing arithmetic progressions $R_1, R_2, \ldots \in \mathrm{Fin}(\mathbb{N})$ of length $1, 2 \ldots$, respectively, such that all sequences in the product $R_1 \times R_2 \times \cdots$ are proper and the partite sumgraph of the sequence $R_1, R_2, \ldots$ is monochromatic.

The above definitions and the Ellis–Numakura Lemma can be extended by replacing the set $\mathbb{N}$ by an infinite set S with an associative operation $+$ on S and treating S as a space with the discrete topology. We will go back to this approach for a moment.

Some combinatorial covering properties have characterizations engaging colorings. For example, this is the case for the Menger property. A cover of a space is a *λ-cover* if it is infinite and each element from the space belongs to infinitely many sets from the cover. Let X be a separable metrizable space with a topology τ. The space X is Menger if and only if for every coloring of $[\tau]^2$ and open ω-cover $\mathcal{U}$ of X, there are pairwise disjoint sets $\mathcal{F}_1, \mathcal{F}_2, \ldots \in \mathrm{Fin}(\mathcal{U})$, such that the family $\bigcup_{n\in\omega} \mathcal{F}_n$ is a λ-cover of X and the partite graph of the sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ is monochromatic ([48, Theorem 10], [28, Theorem 6.2]). Tsaban was the first who proved a result about colorings of a semigroup being a topology of a space with the semigroup operation $\cup$ [65, Theorem 4.6]. Let X be a Menger space with a topology τ. Then for every infinite open cover $\mathcal{U}$ of X and coloring of $[\tau]^2$, there are pairwise disjoint sets $\mathcal{F}_1, \mathcal{F}_2, \ldots \subseteq \mathcal{U}$, such that the family $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is a λ-cover of X, the sequence $\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \ldots$ is proper and the sumgraph of the sequence $\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \ldots$ is monochromatic. Note that the

Milliken–Taylor Theorem follows from the Tsaban Theorem [65, Example 4.7]. One of the key tools, used by Tsaban, are topological games.

For nonempty families of sets $\mathcal{A}$ and $\mathcal{B}$, define a game $\mathsf{G}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ with two players Alice and Bob[2]. In the first round Alice plays a set $A_1 \in \mathcal{A}$ and Bob replies with a finite set $F_1 \subseteq A_1$. In the second round Alice plays a set $A_2 \in \mathcal{A}$ and Bob replies with a finite set $F_2 \subseteq A_2$, etc. During a single play, the following sequence is created

$$(A_1, F_1, A_2, F_2, \dots),$$

where $A_n \in \mathcal{A}$ and F_n is a finite subset of A_n for all natural numbers n. Bob wins the game, if the union $\bigcup_{n\in\mathbb{N}} F_n$ belongs to the family $\mathcal{B}$ and Alice wins, otherwise. If Alice has no winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$, then the statement $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ holds. Sometimes also the inverse implication holds. Hurewicz showed that a space X is Menger $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ if and only if Alice has no winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ played on X [26]. A similar situation appears also for another combinatorial covering properties and games which are associated with these properties. Note that Bob has a winning strategy in the game $\mathsf{G}_{\mathrm{fin}}([\mathbb{N}]^\infty,[\mathbb{N}]^\infty)$, which is a crucial observation for deducing from the below Theorem results about colorings related to natural numbers.

For an infinite set S, let $[S]^\infty$ be the family of all infinite subsets of S.

Theorem 3.14 ([55, Theorem 2.2])**.** *Let S be a semigroup. Assume that a family $\mathcal{A} \subseteq [S]^\infty$ contains an idempotent large for a sequence $\mathcal{R}_1, \mathcal{R}_2, \dots \subseteq \mathrm{Fin}(S)$ and $\mathcal{B} \subseteq [S]^\infty$ is a family such that Alice has no winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$. Then for each coloring of $[S]^2$, there are finite families $\mathcal{F}_1 \subseteq \mathcal{R}_1, \mathcal{F}_2 \subseteq \mathcal{R}_2, \dots$ and sets $F_1 \subseteq \bigcup \mathcal{F}_1, F_2 \subseteq \bigcup \mathcal{F}_2, \dots$ with the following properties.*

(1) The set $\bigcup_{n\in\mathbb{N}} F_n$ is in $\mathcal{B}$.
(2) All sequences in the product $\bigcup \mathcal{F}_1 \times \bigcup \mathcal{F}_2 \times \cdots$ are proper.
(3) The partite sumgraph of the sequence $\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \dots$ is monochromatic.

Theorem 3.14 has many applications which were presented in the work about this topic [55]. We provide here two examples among these applications. The first one is the above Theorem due to Bergelson and Hindman [11], which follows from Theorem 3.14. In order to see this, assume that our semigroup S is the set $\mathbb{N}$ with the usual addition $+$, $\mathcal{A} = \mathcal{B} = [\mathbb{N}]^\infty$ and $\mathcal{R}_1, \mathcal{R}_2, \dots \subseteq \mathrm{Fin}(\mathbb{N})$ is a sequence of families of all increasing arithmetic progressions of length $1, 2, \dots$, respectively and fix a coloring of $[\mathbb{N}]^2$. The family $[\mathbb{N}]^\infty$ contains an idempotent large for the sequence $\mathcal{R}_1, \mathcal{R}_2, \dots$ [19, Lemma 2]. Since Bob has a winning strategy in the game $\mathsf{G}_{\mathrm{fin}}([\mathbb{N}]^\infty,[\mathbb{N}]^\infty)$, the assumptions of Theorem 3.14 are satisfied. Then the sets $\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \dots$ from the claim of Theorem 3.14 contain arithmetic progressions $R_1, R_2, \dots$ of length $1, 2, \dots$, respectively. In particular, the partite sumgraph of the sequence $R_1, R_2, \dots$ is monochromatic.

Another consequences of Theorem 3.14 is an implication in the above result which characterizes the Menger property using colorings. Let X be a separable metrizable Menger space and $\mathcal{U}$ be an open ω-cover of X. We may assume that the cover $\mathcal{U}$ is countable and thus enumerate its elements $\mathcal{U} = \{\, U_n : n \in \mathbb{N} \,\}$. Let a semigroup S be the set $\mathcal{U}$ with the operation $(U_i, U_j) \mapsto U_{\max\{i,j\}}$, where $i, j \in \mathbb{N}$. Let $\mathcal{A} := \{\, \mathcal{V} \in \Omega : \mathcal{V} \subseteq \mathcal{U} \,\}$ and $\mathcal{B}$ be a family of all open λ-covers of X. For each natural number n, define $\mathcal{R}_n := \{\{U_n\}, \{U_{n+1}\}, \dots\}$. Then the family $\mathcal{A}$ contains an idempotent large for the sequence $\mathcal{R}_1, \mathcal{R}_2, \dots$ [55, Example 3.11(3)]. Let Λ be the family of all open λ-covers of a given space. The Menger property $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ is equivalent to $\mathsf{S}_{\mathrm{fin}}(\Omega,\Lambda)$, and thus Alice has no winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\Omega,\Lambda)$ played on X [49, Theorem 5]. Then the sets $F_1, F_2, \dots \subseteq \mathcal{U}$ from the claim of Theorem 3.14 are finite and pairwise disjoint and the partite sumgraph of the sequence $F_1, F_2, \dots$ is equal to the partite graph of this sequence (what follows from the definition of a semigroup operation).

Theorems proved in different papers by Scheepers and his collaborators have the following structure. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty families of sets. Then the property $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ ($\mathsf{S}_1(\mathcal{A},\mathcal{B})$) is equivalent to the following assertion: for every coloring of $[\bigcup \mathcal{A}]^2$ and set $A \in \mathcal{A}$, there are pairwise disjoint finite sets $F_1, F_2, \dots \subseteq A$ (an infinite set $F \subseteq A$), such that the partite graph of the sequence $F_1, F_2, \dots$ (the graph $[F]^2$) is monochromatic. This is the case for covering properties: $\mathsf{S}_{\mathrm{fin}}(\Omega,\Lambda)$ (which is equivalent to the Menger property), $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$, $\mathsf{S}_1(\Omega,\Gamma)$, $\mathsf{S}_1(\Omega,\Omega)$, $\mathsf{S}_1(\Omega,\Lambda)$ (which is equivalent to the Rothberger property) and local properties of $\mathrm{C_p}(X)$, where X is a separable metrizable space: countable fan tightness $\mathsf{S}_{\mathrm{fin}}(\Omega_{\mathbf{0}},\Omega_{\mathbf{0}})$, *countable strong fan tightness* $\mathsf{S}_1(\Omega_{\mathbf{0}},\Omega_{\mathbf{0}})$ and the *strongly Frechet–Urysohn* property $\mathsf{S}_1(\Omega_{\mathbf{0}},\Gamma_{\mathbf{0}})$ (where $\Gamma_{\mathbf{0}} := \{\, A \subseteq \mathrm{C_p}(X) : (\exists f_1, f_2, \dots \in A)(\lim_{n\to\infty} f_n = \mathbf{0}) \,\}$). In each of these results, one of the implications follows from Theorem 3.14 (or from its modification for the game

[2]In a similar manner we define a game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ which is a modification of the game $\mathsf{G}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$, where the sets chosen by Bob are singletons.

$\mathsf{G}_1(\mathcal{A},\mathcal{B})$ [55, Theorem 2.2]). In the paper [55] also uniform proofs of the inverse implications in these results, have been provided.

Main achievements

- Theorems concerning colorings of edge sets of complete graphs with vertices in infinite semigroups. Obtained results are common generalizations for theorems proved in different fields:
  - for natural numbers due to Milliken–Taylor, Deuber–Hindman and Bergelson–Hindman,
  - for combinatorial covering properties due to Scheepers and Tsaban,
  - for local properties of function spaces due to Scheepers.

References

[1] A. Arkhangel'skiĭ, *Hurewicz spaces, analytic sets and fan tightness of function spaces,* Dokl. Akad. Nauk SSSR **287** (1986), 525–528.

[2] L. Aurichi, *D-spaces, topological games, and selection principles*, Topology Proceedings **36** (2010), 107–122.

[3] L. Aurichi, F. Tall, *Lindelöf spaces which are indestructible, productive, or D*, Topology and its Applications **159** (2012), 331–340.

[4] T. Banakh, L. Zdomskyy, *The Coherence of Semifilters: a Survey*, in: **Selection Principles and Covering Properties in Topology** (L. Kočinac, ed.), Quaderni di Matematica **18**, Seconda Universita di Napoli, Caserta, 2006, 53–99.

[5] T. Bartoszyński, H. Judah, *Set Theory: On the structure of the real line*, A. K. Peters, Massachusetts (1995).

[6] T. Bartoszyński, I. Recław, *Not every γ-set is strongly meager*, Contemporary Mathematics **192** (1996), 25–29.

[7] T. Bartoszyński, S. Shelah, *Continuous images of sets of reals*, Topology and its Applications **116** (2001), 243–253.

[8] T. Bartoszyński, S. Shelah, B. Tsaban, *Additivity Properties of Topological Diagonalizations*, Journal of Symbolic Logic **68** (2003), 1254–1260.

[9] T. Bartoszyński, B. Tsaban, *Hereditary topological diagonalizations and the Menger-Hurewicz Conjectures*, Proceedings of the American Mathematical Society **134** (2006), 605–615.

[10] J. Baumgartner, R. Laver, *Iterated perfect set forcing*, Annals of Mathematical Logic **17** (1979), 271–288.

[11] V. Bergelson, N. Hindman, *Ultrafilters and multidimensional Ramsey theorems*, Combinatorica **9** (1989), 1–7.

[12] A. Blass, *Combinatorial cardinal characteristics of the continuum*, in: **Handbook of Set Theory** (M. Foreman, A. Kanamori, eds.), Springer, 2010, 395–489.

[13] A. Blass, S. Shelah, *Ultrafilters with small generating sets*, Israel Journal of Mathematics **65** (1989), 259–271.

[14] L. Bukovský, *Selection principle* S_1 *and combinatorics of open covers*, Topology and its Applications **258** (2019), 239–250.

[15] L. Bukovský, J. Haleš, *QN-spaces, wQN-spaces and covering properties*, Topology and its Applications **154** (2007), 848–858.

[16] L. Bukovský, *On* wQN_* *and* wQN^* *spaces*, Topology and its Applications **156** (2008), 24–27.

[17] L. Bukovský, I. Recław, M. Repický, *Spaces not distinguishing convergences of real-valued functions*, Topology and its Applications **112** (2001), 13–40.

[18] D. Chodounsky, D. Repovš, L. Zdomskyy, *Mathias forcing and combinatorial covering properties of filters*, Journal of Symbolic Logic, **80** (2015), 1398–1410.

[19] W. Deuber, N. Hindman, *Partitions and sums of (m,p,c)-sets*, Journal of Combinatorial Theory, Series A **45** (1987), 300–302.

[20] R. Ellis, *Distal transformation groups*, Pacific Journal of Mathematics **8** (1958), 401–405.

[21] D. Fremlin, A. Miller, *On some properties of Hurewicz, Menger and Rothberger*, Fundamenta Mathematicae **129** (1988), 17–33.

[22] F. Galvin, A. Miller, *γ-sets and other singular sets of real numbers*, Topology and its Applications **17** (1984), 145–155.

[23] J. Gerlits, Zs. Nagy, *Some properties of $C(X)$, I,* Topology and its Applications **14** (1982), 151–161.

[24] J. Haleš, *On Scheepers' conjecture*, Acta Universitatis Carolinae. Mathematica et Physica **46** (2005), 27–31.

[25] N. Hindman, *Finite sums from sequences within cells of a partition of* $\mathbb{N}$, Journal of Combinatorial Theory, Series A **17** (1974), 1–11.

[26] W. Hurewicz, *Über eine Verallgemeinerung des Borelschen Theorems*, Mathematische Zeitschrift **24** (1925), 401–421.

[27] W. Hurewicz, *Über Folgen stetiger Funktionen*, Fundamenta Mathematicae **9** (1927), 193–204.

[28] W. Just, A. Miller, M. Scheepers, P. Szeptycki, *The combinatorics of open covers II*, Topology and its Applications **73** (1996), 241–266.

[29] A. Krawczyk, H. Michalewski, *Linear metric spaces close to being σ-compact*, Technical Report 46 (2001) of the Institute of Mathematics, Warsaw University.

[30] M. Krupski, *Games and hereditary Baireness in hyperspaces and spaces of probability measures*, Journal of the Institute of Mathematics of Jussieu **21** (2022), 851–868..

[31] R. Laver, *On the consistency of Borel's conjecture*, Acta Mathematicae **137** (1976), 151–169.

[32] K. Menger, *Einige Überdeckungssätze der Punktmengenlehre*, Sitzungsberichte der Wiener Akademie **133** (1924), 421–444.

[33] A. Miller, *A nonhereditary Borel-cover γ-set*, Real Analysis Exchange **29** (2003), 601 – 606.

[34] A. Miller, B. Tsaban, *Point-cofinite covers in Laver's model*, Proceedings of the American Mathematical Society **138** (2010) 3313–3321.

[35] A. Miller, B. Tsaban, L. Zdomskyy, *Selective covering properties of product spaces*, Annals of Pure and Applied Logic **165** (2014), 1034–1057.

[36] A. Miller, B. Tsaban, L. Zdomskyy, *Selective covering properties of product spaces, II: γ spaces*, Transactions of the American Mathematical Society **368** (2016), 2865–2889.
[37] K. Milliken, *Ramsey's theorem with sums or unions*, Journal of Combinatorial Theory, Series A **18** (1975), 276–290.
[38] K. Numakura, *On bicompact semigroups*, Mathematical Journal of Okayama University **1** (1952), 99–108.
[39] T. Orenshtein, B. Tsaban, *Linear σ-additivity and some applications*, Transactions of the American Mathematical Society **363** (2011), 3621–3637.
[40] A. Osipov, P. Szewczak, B. Tsaban, *Strongly sequentially separable function spaces, via selection principles*, Topology and its Applications, **270** (2020), 106942.
[41] I. Recław, *Every Luzin set is undetermined in the point-opened game*, Fundamenta Mathematicae **144** (1994), 43–54.
[42] D. Repovš, L. Zdomskyy, *On M-separability of countable spaces and function spaces*, Topology and its Applications **157** (2010), 2538–2541.
[43] F. Rothberger, *Eine Verschärfung der Eigenschaft C*, Fundamenta Mathematicae **30** (1938), 50–55.
[44] M. Sakai, *Property C" and function spaces*, Proceedings of the American Mathematical Society **104** (1988), 917–919.
[45] M. Sakai, *The sequence selection properties of* $\mathrm{C_p}(X)$, Topology and its Applications **154** (2007), 552–560.
[46] M. Sakai, *Selection principles and upper semicontinuous functions*, Colloquium Mathematicum **117** (2009), 251–256..
[47] G. Sacks, *Forcing with perfect closed sets*, in: **Axiomatic Set Theory** (D. Scott, ed.), Proceedings of Symposia in Pure Mathematics, Vol. XIII, Part I, American Mathematical Society, Providence, RI, 1971, 331–355.
[48] M. Scheepers, *Combinatorics of open covers. I. Ramsey theory*, Topology and its Applications **69** (1996), 31–62.
[49] M. Scheepers, *Open covers and partition relations*, Proceedings of the American Mathematical Society **127** (1999), 577–581.
[50] M. Scheepers, *Sequential convergence in* $\mathrm{C_p}(X)$ *and a covering property*, East-West Journal of Mathematics **1** (1999), 207–214.
[51] M. Scheepers, *The length of some diagonalization games*, Archive for Mathematical Logic **38** (1999), 103–122.
[52] M. Scheepers, *Combinatorics of open covers, VI. Selectors for sequences of dense sets*, Quaestiones Mathematicae **22** (1999), 109–130.
[53] M. Scheepers, B. Tsaban, *The combinatorics of Borel covers*, Topology and its Applications **121** (2002), 357–382.
[54] S. Shelah, *Every null-additive set is meager-additive*, Israel Journal of Mathematics **89** (1995), 357–376.
[55] P. Szewczak, *Abstract colorings, games and ultrafilters*, Topology and its Applications **335** (2023), 108595.
[56] P. Szewczak, *Productivity of paracompactness and closed images of real GO-spaces*, Topology and its Applications **222** (2017), 254–273.
[57] P. Szewczak, B. Tsaban, *Products of Menger spaces: a combinatorial aproach*, Annals of Pure and Applied Logic **168** (2017), 1–18.
[58] P. Szewczak, B. Tsaban, *Products of general Menger spaces*, Topology and its Applications **255** (2019), 41–55.
[59] P. Szewczak, B. Tsaban, L. Zdomskyy, *Finite powers and products of Menger sets*, Fundamenta Mathematicae **253** (2021), 257–275.
[60] P. Szewczak, T. Weiss, *Null sets and combinatorial covering properties*, Journal of Symbolic Logic, (2021), 1–11, doi:10.1017/jsl.2021.51.
[61] P. Szewczak, G. Wiśniewski, *Products of Luzin-type sets with combinatorial properties*, Topology and its Applications **264** (2019), 420–433.
[62] P. Szewczak, M. Włudecka, *Unbounded towers and products*, Annals of Pure and Applied Logic **172** (2021), 102900.
[63] A. Taylor, *A canonical partition relation for finite subsets of* ω, Journal of Combinatorial Theory, Series A **21** (1976), 137–146.
[64] B. Tsaban, *Additivity numbers of covering properties*, in: **Selection Principles and Covering Properties in Topology** (L. Kocinac, editor), Quaderni di Matematica 18, Seconda Universita di Napoli, Caserta 2006, 245–282.
[65] B. Tsaban, *Algebra, selections, and additive Ramsey theory*, Fundamenta Mathematicae **240** (2018), 81–104.
[66] B. Tsaban, L. Zdomskyy, *Combinatorial images of sets of reals and semifilter trichotomy*, Journal of Symbolic Logic **73** (2008), 1278–1288.
[67] B. Tsaban, L. Zdomskyy, *Hereditarily Hurewicz spaces and Arhangel'skiĭ sheaf amalgamations*, Journal of the European Mathematical Society **12** (2012), 353–372.
[68] S. Todorčević, *Aronszajn orderings*, Publications de l'Insitut Mathematique **57** (1995), 29–46.
[69] B. Tsaban, *Selection Principles and special sets of reals*, in: **Open Problems in Topology II** (E. Pearl, ed.), Elsevier B.V. 2007, 91–108.
[70] B. Tsaban, *Menger's and Hurewicz's Problems: Solutions from "The Book" and refinements*, Contemporary Mathematics **533** (2011), 211–226.
[71] B. Tsaban, L. Zdomskyy, *Scales, fields, and a problem of Hurewicz*, Journal of the European Mathematical Society **10** (2008), 837–866.
[72] L. Zdomskyy, *A semifilter approach to selection principles*, Commentationes Mathematicae Universitatis Carolinae **46** (2005), 525–539.
[73] L. Zdomskyy, *Products of Menger spaces in the Miller model*, Advances in Mathematics **335** (2018), 170–179.
[74] O. Zindulka, *Strong measure zero and the like via selection principles*, Conference Frontiers of Selection Principles, Poland, 2017, plenary lecture, slides available at: `http://selectionprinciples.com/Talks/Zindulka.pdf`.

4. Scientific activity

Grants.

Project investigator of a Polish team .... 08.2022 – 07.2025

Weave–Unisono Grant, National Science Center, Poland/Austrian Science Fund (FWF)

Managing a three-person research team of the Polish part of the project under a three-year bilateral grant implemented with the Vienna University of Technology. Project investigator of an Austrian team: prof. Lyubomyr Zdomskyy.
Project: *Set-theoretic aspects of topological selections.*
Project number: 2021/03/Y/ST1/00122
Funds of the Polish part: 955 458 PLN.

Invited lectures.

Boise Extravaganza in Set Theory 06.2019
Ashland, Oregon, USA
Plenary Lecture: *Selection principles in mathematics*

Frontiers of Selection Principles 08.2017
Warsaw, Poland
Series of 5 lectures: *Introduction to selection principles*
Plenary lecture: *Products of Menger spaces and the Scheepers property*

Workshop on Selection Principles in Mathematics 09.2016
Chengdu, China
Series of 5 lectures: *Selection principles in mathematics*

Internships

1-year post-doctoral internship .... 02.2016 – 01.2017
Bar-Ilan University, Israel
1-year post-doctoral internship under the Kolman–Soref grant under supervision of prof. Boaz Tsaban.
Project: *Coherent omission of intervals.*

1-month internship 10.2016
Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Austria
1-month internship under supervision of prof. Lyubomyr Zdomskyy.
Project: *Products of Menger sets.*

5. Organizational achievements

Organizing committee 08.2017
Frontiers of Selection Principles, Warsaw, Poland
Organizing committee (together with prof. Boaz Tsaban and prof. Lyubomyr Zdomskyy). The conference was fully dedicated to combinatorial covering properties and it gathered 60 researchers from 15 countries. The conference was preceded by a 1-week workshop for students and young researchers.

Editorial board 2017–2020
Journal: Topology and its Applications
Publisher: Elsevier
Editorial board (together with prof. Boaz Tsaban and prof. Lyubomyr Zdomskyy) related to the special issue of *Topology and its Applications* dedicated to the conference *Frontiers of Selection Principles.*
Link: `https://www.sciencedirect.com/journal/topology-and-its-applications/special-issue/108JC10LKMT`.