STRONGLY SEQUENTIALLY SEPARABLE FUNCTION SPACES, VIA SELECTION PRINCIPLES

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Abstract. A separable space is *strongly sequentially separable* if, for each countable dense set, every point in the space is a limit of a sequence from the dense set. We consider this and related properties, for the spaces of continous and Borel real-valued functions on Tychonoff spaces, with the topology of pointwise convergence. Our results solve a problem stated by Gartside, Lo, and Marsh.

1. Introduction

We apply methods of selection principles to a problem of Gartside, Lo, and Marsh [6, Problem 19].

By *space* we mean a topological space. A space is *Fréchet–Urysohn* if each point in the closure of a set is a limit of a sequence from the set. A separable space is *strongly sequentially separable (SSS)* [9] if, for each countable dense set, every point in the space is a limit of a sequence from the dense set.

Every Fréchet–Urysohn space is strongly sequentially separable, but not conversely [1, Example 2.4]. For a Tychonoff space X, let $\mathrm{C}(X)$ and $\mathrm{B}(X)$ be the spaces of continuous and Borel, respectively, real-valued functions on X, with the topology of pointwise convergence. We are only concerned with uncountable spaces. In this case, the space $\mathrm{B}(X)$ is never Fréchet–Urysohn. Strong sequential separability is hereditary for dense subspaces, and we have the following implications.

$$\begin{array}{ccccc} \mathbb{R}^X \text{ is SSS} & \longrightarrow & \mathrm{B}(X) \text{ is SSS} & \longrightarrow & \mathrm{C}(X) \text{ is SSS} \\ & & & & \uparrow \\ & & & & \mathrm{C}(X) \text{ is Fréchet–Urysohn} \end{array}$$

It is consistent that the properties in this diagram hold only for countable spaces X and are, thus, equivalent [6, Corollary 17]. This motivates the following problem [6, Problem 19].

Problem 1 (Gartside–Lo–Marsh)**.** *Is there, consistently, a Tychonoff space X such that the space $\mathrm{C}(X)$ is strongly sequentially separable but not Fréchet–Urysohn, and the space $\mathbb{R}^X$ is not strongly sequentially separable?*

We solve this problem, and all other problems suggested by the above diagram. To this end, we extend Arhangel'skiĭ's local-to-global duality, dualize these problems to ones concerning covering properties, and apply the theory of selection principles.

2010 *Mathematics Subject Classification.* 37F20, 26A03, 03E75, 54C35.

Key words and phrases. strong sequential separability, function spaces, selection principles, Gerlits–Nagy, γ-property, Borel function, $\binom{\Omega}{\Gamma}$, $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$, γ-set, C-space.

2. Local-to-global Duality

A *cover* of a space is a family of *proper* subsets whose union is the entire space. For families $\mathbf{A}$ and $\mathbf{B}$ of covers of a space, the property that every cover in the family $\mathbf{A}$ has a subcover in the family $\mathbf{B}$ is denoted $\binom{\mathbf{A}}{\mathbf{B}}$. An *$\omega$-cover* is a cover such that each finite subset of the space is contained in some set from the cover. A *γ-cover* is an infinite cover such that each point of the space belongs to all but finitely many sets from the cover.

An *open cover* is a cover by open sets. Similarly, we define *Borel cover*, *clopen cover*, etc. Given a space, let Ω, Ω_{ctbl}, Ω_{Bor} and Γ, be the families of *open* ω-covers, *countable open* ω-covers, *countable Borel* ω-covers, and γ-covers, respectively.

The property $\binom{\Omega}{\Gamma}$ is the celebrated γ-property of Gerlits and Nagy, who proved that a Tychonoff space has this property if and only if the space $\mathrm{C}(X)$ is Fréchet–Urysohn [7, Theorem 2].

Lemma 2. *For each space X, the following assertions are equivalent:*

1. *The space* $\mathrm{B}(X)$ *is strongly sequentially separable.*
2. *The space X has a coarser second countable topology and it satisfies* $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$.

Proof. (1) $\Rightarrow$ (2): Since the space $\mathrm{B}(X)$ is separable, the space X has a coarser second countable topology [15, Theorem 1].

Let $\mathcal{U} \in \Omega_{\mathrm{Bor}}(X)$, and H be a countable dense subset of the space $\mathrm{B}(X)$. For a Borel set $U \subseteq X$ and a function $h \in H$, let $f_{U,h} \in \mathrm{B}(X)$ be the function such that $f_{U,h} \restriction U := h \restriction U$ and $f_{U,h} \restriction (X \setminus U) := 1$. The set $D := \{ f_{U,h} : U \in \mathcal{U}, h \in H \}$ is a countable dense subset of $\mathrm{B}(X)$. By (1), there is a sequence $\{ f_{U_n,h_n} : n \in \mathbb{N} \}$ in the set D, converging to the zero function $\mathbf{0}$. Let F be a finite subset of X. The set $W := \{ f \in \mathrm{B}(X) : f[F] \subseteq (-1,1) \}$ is a neighborhood of $\mathbf{0}$ in $\mathrm{B}(X)$. For a natural number n, if $f_{U_n,h_n} \in W$, then $F \subseteq U_n$. Since all but finitely many elements of the sequence belong to the set W, we have $\{ U_n : n \in \mathbb{N} \} \in \Gamma(X)$. Thus, the space X satisfies $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$.

(2) $\Rightarrow$ (1): The property $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ implies that every point in the closure of a countable set in $\mathrm{B}(X)$ is the limit of a sequence from that set [12, Lemma 2.8]. $\square$

Let $\mathbb{N}$ be the set of natural numbers. For infinite sets $a, b \subseteq \mathbb{N}$ we write $a \subseteq^* b$ if the set $a \setminus b$ is finite. A *pseudointersection* of a family of infinite sets is an infinite set a with $a \subseteq^* b$ for all sets b in the family. A subfamily of $[\mathbb{N}]^\infty$ is *centered* if the finite intersections of its elements, are infinite. Let $\mathfrak{p}$ be the minimal cardinality of a family of infinite subsets of $\mathbb{N}$ that is centered and has no pseudointersection. Information about the cardinal number $\mathfrak{p}$ is available, for example, in van Douwen's survey [4]. Gartside, Lo and Marsh proved that a Tychonoff product $\mathbb{R}^X$ is strongly sequentially separable if and only if $|X| < \mathfrak{p}$ [6, Theorem 11].

Gartside, Lo and Marsh proved that a function space $\mathrm{C}(X)$ is strongly sequentially separable if and only if the space X has a coarser second countable topology, and every coarser second countable topology for X satisfies $\binom{\Omega}{\Gamma}$ [6, Theorem 16]. The property $\binom{\Omega}{\Gamma}$ implies $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$, and we have the following observation.

Lemma 3. *If a space X has a coarser second countable topology, then the following assertions are equivalent:*

1. $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$.
2. *Every coarser second countable topology for the space X has the property* $\binom{\Omega}{\Gamma}$.

Proof. (1) ⇒ (2): Second countable spaces have the property $\binom{\Omega}{\Omega_{\mathrm{ctbl}}}$; every open ω-cover is refined by an ω-cover consisting of finite unions of basic open sets.

(2) ⇒ (1): Let $\mathcal{U}$ be a countable open ω-cover of X, and $\mathcal{B}$ be a countable base for a coarser topology for X. The family $\mathcal{B} \cup \mathcal{U}$ generates a second countable topology on the set X, and the open ω-cover $\mathcal{U}$ has a subcover in Γ. □

In summary, for spaces X with a coarser second countable topology, the diagram from the previous section dualizes to the following one.

$$\begin{array}{ccccc} |X| < \mathfrak{p} & \longrightarrow & X \text{ satisfies } \binom{\Omega_{\mathrm{Bor}}}{\Gamma} & \longrightarrow & X \text{ satisfies } \binom{\Omega_{\mathrm{ctbl}}}{\Gamma} \\ & & & & \uparrow \\ & & & & X \text{ satisfies } \binom{\Omega}{\Gamma} \end{array}$$

Problem 1 is thus reduced to the following problem.

Problem 4. *Is there, consistently, a Tychonoff space X with a coarser second countable topology, that satisfies $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$ but not $\binom{\Omega}{\Gamma}$, with $|X| \geq \mathfrak{p}$?*

We will solve this problem, as well as its variations.

3. The problems and their solutions

Consider the positions in the diagrams from the previous section. Write there "$\bullet$" if the property holds, and "$\circ$" if it does not. For example, sets $X \subseteq \mathbb{R}$ of cardinality smaller than $\mathfrak{p}$ realize the following setting.

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & & & \uparrow \\ & & & & \bullet \end{array},$$

that will be denoted $\bullet\,\bullet\,\underset{\bullet}{\bullet}$ for brevity. We consider the consistency of all settings that are not ruled out by the implications in the diagram. These are the following settings:

$$\circ\,\circ\,\underset{\circ}{\circ} \qquad \circ\,\circ\,\underset{\circ}{\bullet} \qquad \circ\,\bullet\,\underset{\circ}{\bullet} \qquad \circ\,\circ\,\underset{\bullet}{\bullet} \qquad \bullet\,\bullet\,\underset{\circ}{\bullet} \qquad \circ\,\bullet\,\underset{\bullet}{\bullet} \qquad \bullet\,\bullet\,\underset{\bullet}{\bullet}$$

Problem 1 asks whether either of the the settings $\circ\,\bullet\,\underset{\circ}{\bullet}$ or $\circ\,\circ\,\underset{\bullet}{\bullet}$ is consistent.

The hypothesis $\aleph_1 < \mathfrak{p}$ and its negation ($\aleph_1 = \mathfrak{p}$) are both consistent [4, Theorem 5.1]. The following proposition is a variation of an earlier result [14, Example 4.7].

Proposition 5 ($\bullet\,\bullet\,\underset{\circ}{\bullet}$)**.** *The following assertions are equivalent.*

1. *There is a Tychonoff space X such that the space $\mathbb{R}^X$ is strongly sequentially separable, but the space $\mathrm{C}(X)$ is not Fréchet–Urysohn.*
2. $\aleph_1 < \mathfrak{p}$.

Proof. Recall that $\mathbb{R}^X$ is strongly sequentially separable if and only if $|X| < \mathfrak{p}$.

(1) ⇒ (2): The given space X has $|X| < \mathfrak{p}$. Had it been countable, the space $\mathrm{C}(X)$ would have been metrizable.

(2) ⇒ (1): A discrete space of cardinality $\aleph_1$ is not Lindelöf, and thus does not satisfy $\binom{\Omega}{\Gamma}$. Apply duality. □

The following folklore fact implies that discrete spaces of cardinality $\mathfrak{p}$ or greater have none of the studied properties.

Lemma 6 ($\circ\circ{}^{\circ}_{\circ}$)**.** *A discrete space X is $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$ if and only if $|X| < \mathfrak{p}$.*

Proof. ($\Leftarrow$) Let $\{\, U_n : n \in \mathbb{N} \,\} \in \Omega_{\mathrm{ctbl}}$. For each element $x \in X$, let $a_x := \{\, n \in \mathbb{N} : x \in U_n \,\}$, an infinite subset of $\mathbb{N}$. Since $|X| < \mathfrak{p}$, the family $\{\, a_x : x \in X \,\}$ has a pseudointersection a. Then $\{\, U_n : n \in a \,\} \in \Gamma$.

($\Rightarrow$) Assume that $|X| \geq \mathfrak{p}$. Since the space X is discrete, we may assume that it is a family of infinite subsets of $\mathbb{N}$ of cardinality $\mathfrak{p}$, that is centered and has no pseudointersection, equipped with the discrete topology. The family of (open) sets $\{\, U_n : n \in \mathbb{N} \,\}$, defined by $U_n := \{\, x \in X : n \in x \,\}$ for natural numbers n, is in Ω_{ctbl} and has no subfamily in Γ. □

A theorem of Galvin and Miller [5, Theorem 2] asserts that, if $\mathfrak{p} = |\mathbb{R}|$, then there is a set $X \subseteq \mathbb{R}$ of cardinality $\mathfrak{p}$, satisfying $\binom{\Omega}{\Gamma}$. The Galvin–Miller Theorem is refined by Theorem 7 of Orenshtein and Tsaban [13, Theorem 3.6]. Since this result is central to the remainder of this paper, we include here a simpler proof, due to the third named author [19].

We identify the Cantor space $\{0,1\}^{\mathbb{N}}$ with the family $\mathrm{P}(\mathbb{N})$ of all subsets of the set $\mathbb{N}$. Thus, we view the space $\mathrm{P}(\mathbb{N})$ as a subset of the real line. The space $\mathrm{P}(\mathbb{N})$ splits into two subspaces: the family of infinite subsets of $\mathbb{N}$, denoted $[\mathbb{N}]^\infty$, and the family of finite subsets of $\mathbb{N}$, denoted Fin. We identify every set $a \in [\mathbb{N}]^\infty$ with its increasing enumeration, an element of the Baire space $\mathbb{N}^{\mathbb{N}}$. Thus, for a natural number n, $a(n)$ is the n-th element in the increasing enumeration of the set a. This way, we have $[\mathbb{N}]^\infty \subseteq \mathbb{N}^{\mathbb{N}}$, and the topology of the space $[\mathbb{N}]^\infty$ (a subspace of the Cantor space $\mathrm{P}(\mathbb{N})$) coincides with the subspace topology induced by $\mathbb{N}^{\mathbb{N}}$. When an element of $[\mathbb{N}]^\infty$ is viewed as an element of $\mathbb{N}^{\mathbb{N}}$, we refer to it as a *function*.

For functions $a, b \in [\mathbb{N}]^\infty$, we write $a \leq^* b$ if the set $\{\, n : b(n) < a(n) \,\}$ is finite. Let $A \subseteq [\mathbb{N}]^\infty$. For a function $b \in [\mathbb{N}]^\infty$, we write $A \leq^* b$ if $a \leq^* b$ for all functions $a \in A$. The set A is *unbounded* if there is no function $b \in [\mathbb{N}]^\infty$ with $A \leq^* b$. Let $\mathfrak{b}$ be the minimal cardinality of an unbounded set in $[\mathbb{N}]^\infty$. A set $\{\, x_\alpha : \alpha < \mathfrak{b} \,\} \subseteq [\mathbb{N}]^\infty$ is an *unbounded tower* if it is unbounded and for all ordinal numbers $\alpha, \beta < \mathfrak{b}$ with $\alpha < \beta$, we have $x_\alpha \,{}^*\!\supseteq x_\beta$. An unbounded tower of cardinality $\mathfrak{p}$ exists if (and only if) $\mathfrak{p} = \mathfrak{b}$ [13, Lemma 3.3].

Theorem 7 (Orenshtein–Tsaban)**.** *For each unbounded tower $T \subseteq [\mathbb{N}]^\infty$ of cardinality $\mathfrak{p}$, the set $T \cup \mathrm{Fin}$ of real numbers satisfies $\binom{\Omega}{\Gamma}$.*

In order to prove Theorem 7, we need the following notions and auxiliary results. Let n, m be natural numbers with $n < m$. Define $(n, m) := \{\, i \in \mathbb{N} : n < i < m \,\}$. A set $a \in [\mathbb{N}]^\infty$ *omits* the interval (n, m) if $a \cap (n, m) = \emptyset$. For a space X, let $\Omega(X)$ be the family of all *open ω-covers* of X, and $\Gamma(X)$ be the family of all *open γ-covers* of X.

Lemma 8 (Galvin–Miller[5, Lemma 1.2])**.** *Let $\mathcal{U}$ be a family of open sets in $\mathrm{P}(\mathbb{N})$ such that $\mathcal{U} \in \Omega(\mathrm{Fin})$. There are a function $b \in [\mathbb{N}]^\infty$ and sets $U_1, U_2, \ldots \in \mathcal{U}$ such that for each element $x \in [\mathbb{N}]^\infty$ and all natural numbers n:*

$$\text{If } x \cap ((b(n), b(n+1)) = \emptyset, \text{ then } x \in U_n.$$

Lemma 9 (Folklore [18, Lemma 2.13])**.** *Let Y be a subset of $[\mathbb{N}]^\infty$. The set Y is unbounded if and only if, for each function $b \in [\mathbb{N}]^\infty$, there is a set $a \in Y$ that omits infinitely many intervals $(b(n), b(n+1))$.*

Lemma 10. *Let $X \subseteq \mathrm{P}(\mathbb{N})$ be a set such that $\mathrm{Fin} \subseteq X$ and $|X| < \mathfrak{p}$. Let $\mathcal{U}$ be a family of open sets in $\mathrm{P}(\mathbb{N})$ such that $\mathcal{U} \in \Omega(X)$, and Y be an unbounded set in $[\mathbb{N}]^\infty$. There are a* *set $a \in Y$, and sets $U_1, U_2, \ldots \in \mathcal{U}$ such that $\{ U_n : n \in \mathbb{N} \} \in \Gamma(X)$, and for each element $x \in [\mathbb{N}]^\infty$ and all natural numbers n:*

$$\text{If } x \setminus \{ 1, \ldots, n \} \subseteq a, \text{ then } x \in \bigcap_{k \geq n} U_k.$$

Proof. Since $|X| < \mathfrak{p}$, the set X satisfies $\binom{\Omega}{\Gamma}$ [16, Proposition 2]. Let $\mathcal{V} \in \Gamma(X)$ be a subfamily of $\mathcal{U}$. By Lemma 8, there are a function $b \in [\mathbb{N}]^\infty$, and sets $V_1, V_2, \ldots \in \mathcal{V}$ such that for each element $x \in [\mathbb{N}]^\infty$, and all natural numbers i:

$$\text{If } x \cap (b(i), b(i+1)) = \emptyset, \text{ then } x \in V_i. \tag{1}$$

By Lemma 9, there is a set $a \in Y$ such that the set

$$c := \{ i \in \mathbb{N} : a \cap (b(i), b(i+1)) = \emptyset \}$$

is infinite. Fix a natural number n. Let k be a natural number with $n \leq k$, and $x \in [\mathbb{N}]^\infty$ be an element such that $x \setminus \{ 1, \ldots, n \} \subseteq a$. Then $n \leq c(k)$, and we have

$$x \cap (b(c(k)), b(c(k)+1)) \subseteq a \cap (b(c(k)), b(c(k)+1)) = \emptyset.$$

By (1), we have $x \in V_{c(k)}$. Thus, $x \in \bigcap_{k \geq n} V_{c(k)}$.

Since $\mathcal{V} \in \Gamma(X)$, we have $\{ V_{c(i)} : i \in \mathbb{N} \} \in \Gamma(X)$. □

Proof of Theorem 7. Let $\{ x_\alpha : \alpha < \mathfrak{b} \} \subseteq [\mathbb{N}]^\infty$ be an unbounded tower. Let $X := \mathrm{Fin} \cup \{ x_\alpha : \alpha < \mathfrak{b} \}$, and for ordinal numbers $\gamma < \mathfrak{b}$, let $X_\gamma := \mathrm{Fin} \cup \{ x_\alpha : \alpha < \gamma \}$. Let $\mathcal{U} \in \Omega(X)$. Fix an ordinal number $\gamma_0 < \mathfrak{b}$. By induction, for a natural number $m > 0$, we proceed as follows. By Lemma 10, there are an ordinal number $\gamma_m < \mathfrak{b}$, and a subfamily $\{ U^{(m)}_n : n \in \mathbb{N} \} \in \Gamma(X_{\gamma_{m-1}})$ of $\mathcal{U}$ such that, for each element $x \in [\mathbb{N}]^\infty$ and all natural numbers n:

$$\text{If } x \setminus \{ 1, \ldots, n \} \subseteq x_{\gamma_m}, \text{ then } x \in \bigcap_{k \geq n} U^{(m)}_k. \tag{2}$$

Let $\gamma := \sup_n \gamma_n$. There is a function $g \in [\mathbb{N}]^\infty$ such that $x_\gamma \setminus \{ 1, \ldots, g(n) \} \subseteq x_{\gamma_n}$ for all natural numbers n. Fix an ordinal number α with $\gamma \leq \alpha < \mathfrak{b}$. Since $x_\alpha \subseteq^* x_\gamma$, we have

$$x_\alpha \setminus \{ 1, \ldots, g(n) \} \subseteq x_\gamma \setminus \{ 1, \ldots, g(n) \} \subseteq x_{\gamma_n},$$

for all but finitely many natural numbers n. By (2), we have $x_\alpha \in \bigcap_{k \geq g(n)} U^{(n)}_k$ for all but finitely many natural numbers n. Thus, for any function $h \in [\mathbb{N}]^\infty$ with $g \leq^* h$, we have $\{ U^{(n)}_{h(n)} : n \in \mathbb{N} \} \in \Gamma(\{ x_\alpha : \gamma \leq \alpha < \mathfrak{b} \})$.

For each element $x \in X_\gamma$, and each natural number n, define

$$f_x(n) := \min \left\{ m \in \mathbb{N} : x \in \bigcap_{k \geq m} U^{(n)}_k \right\}$$

if the set is nonempty, and $f_x(n) := 0$ otherwise. Since $|X_\gamma| < \mathfrak{b}$, there is a function $h \in [\mathbb{N}]^\infty$ such that $\{ f_x : x \in X_\gamma \} \cup \{ g \} \leq^* h$, and the sets $U^{(n)}_{h(n)}$ are distinct. Then $\{ U^{(n)}_{h(n)} : n \in \mathbb{N} \} \in \Gamma(X_\gamma)$. Since $\{ U^{(n)}_{h(n)} : n \in \mathbb{N} \} \in \Gamma(\{ x_\alpha : \gamma \leq \alpha < \mathfrak{b} \})$ as well, we have $\{ U^{(n)}_{h(n)} : n \in \mathbb{N} \} \in \Gamma(X)$. □

4. Subsets of the Real, Michael, and Sorgenfrey line

The *Michael line* [10] is the set $\mathrm{P}(\mathbb{N})$, with the topology where the points of the set $[\mathbb{N}]^\infty$ are isolated, and the neighborhoods of the points of the set Fin are those induced by the Cantor space topology on $\mathrm{P}(\mathbb{N})$. The *Sorgenfrey line* [17] is the set $\mathbb{R}$ with the topology generated by the half-open intervals $[a,b)$, for $a,b\in\mathbb{R}$.

The forthcoming Theorem 11(2) solves the problem of Gartside–Lo–Marsh Problem (Problem 1). Recall that an unbounded tower in $[\mathbb{N}]^\infty$ of cardinality $\mathfrak{p}$ exists if and only if $\mathfrak{p}=\mathfrak{b}$.

Theorem 11. *Let $T\subseteq[\mathbb{N}]^\infty$ be an unbounded tower of cardinality $\mathfrak{p}$.*

(1) $\left(\circ\,\circ\,\substack{\bullet\\ \bullet}\right)$ *As a subset of $\mathbb{R}$, the set $T\cup\mathrm{Fin}$ satisfies $\binom{\Omega}{\Gamma}$ but not $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$.*
(2) $\left(\circ\,\circ\,\substack{\bullet\\ \circ}\right)$ *Assume that $\aleph_1<\mathfrak{p}$. As a subset of the Michael line, the set $T\cup\mathrm{Fin}$ satisfies $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$ but neither $\binom{\Omega}{\Gamma}$ nor $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$.*

Proof. (1) By Theorem 7, the set $T\cup\mathrm{Fin}$ satisfies $\binom{\Omega}{\Gamma}$. The set T is centered and has no pseudointersection. Since the set T is a Borel subset of $T\cup\mathrm{Fin}$, and the property $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ is hereditary for Borel subsets, the set $T\cup\mathrm{Fin}$ does not satisfy $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$, too.

(2) By Lemma 6, every space of cardinality smaller than $\mathfrak{p}$ satisfies $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$. For a set $U\subseteq\mathrm{P}(\mathbb{N})$, let $\mathrm{Int}(U)$ be the interior of the set U in the Cantor space topology on $\mathrm{P}(\mathbb{N})$. If $\mathcal{U}\in\Omega(\mathrm{Fin})$ is a family of open sets in the Michael line, then $\{\,\mathrm{Int}(U):U\in\mathcal{U}\,\}\in\Omega(\mathrm{Fin})$. Thus, the proof of Theorem 7 actually establishes that the set $T\cup\mathrm{Fin}$, as a subspace of the Michael line, satisfies $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$.

Write $T=\{\,x_\alpha:\alpha<\mathfrak{b}\,\}$ with $x_\alpha\subseteq^* x_\beta$ for $\beta<\alpha$. The set $A:=\{\,x\in T:x_{\omega_1}\subseteq^* x\,\}$ has cardinality $\aleph_1$. The set A is F_σ in the Cantor space topology and, in particular, in the Michael line topology. Thus, the space $T\cup\mathrm{Fin}$ has an uncountable discrete F_σ subset. Since the Lindelöf property is hereditary for F_σ subsets, the space $T\cup\mathrm{Fin}$ is not Lindelöf. Since every space with the property $\binom{\Omega}{\Gamma}$ is Lindelöf, the space $T\cup\mathrm{Fin}$ does not satisfy $\binom{\Omega}{\Gamma}$.

Since every Borel set in the Cantor space is also Borel in the Michael line, by (1) the space $T\cup\mathrm{Fin}$ does not satisfy $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$. □

Assuming the Continuum Hypothesis, there is an uncountable set of real numbers satisfying $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ ([3, Theorem 4.1], [11, Theorem 5]).

Theorem 12. *Let $X\subseteq\mathbb{R}$ be an uncountable set satisfying $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$.*

(1) *As a subset of $\mathbb{R}$, the set $-X\cup X$ satisfies $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ and not $\binom{\Omega}{\Gamma}$.*
(2) *As a subset of the Sorgenfrey line, the set $-X\cup X$ satisfies $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ but not $\binom{\Omega}{\Gamma}$.*

In particular, if the Continuum Hypothesis holds, we obtain the setting $\circ\,\bullet\,\substack{\bullet\\ \bullet}$ from (1), and the setting $\circ\,\bullet\,\substack{\bullet\\ \circ}$ from (2). If $\aleph_1<\mathfrak{p}$, we obtain the settings $\bullet\,\bullet\,\substack{\bullet\\ \bullet}$ and $\bullet\,\bullet\,\substack{\bullet\\ \circ}$.

Proof. Let $Y\subseteq\mathbb{R}$ be an uncountable set satisfying $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$. The disjoint union $Y\sqcup Y$ satisfies $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ as well: Let $\mathcal{U}$ be a countable Borel ω-cover of $Y\sqcup Y$. The family

$$\mathcal{V}:=\{\,U\cap V:U\sqcup V\in\mathcal{U},\ U\subseteq Y\sqcup\emptyset, V\subseteq\emptyset\sqcup Y\,\}$$

is a countable Borel ω-cover of Y. Let $\mathcal{W}\subseteq\mathcal{V}$ be a γ-cover of Y. Then the family

$$\{\,U\sqcup V\in\mathcal{U}:U\cap V\in\mathcal{W},\ U\subseteq Y\sqcup\emptyset, V\subseteq\emptyset\sqcup Y\,\}$$

is a γ-cover of $Y\sqcup Y$.

The set $X := Y \cup \{ -y : y \in Y \}$, a continuous image of the space $Y \sqcup Y$, satisfies $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$, too. Consider this set as a subspace of the Sorgenfrey line. Since the Borel sets in the real line and the Sorgenfrey line are the same, the space X satisfies $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$.

The product space $X \times X$ contains the uncountable closed discrete set $\{ (x, -x) : x \in X \}$, and thus does not satisfy $\binom{\Omega}{\Gamma}$. The property $\binom{\Omega}{\Gamma}$ is preserved by finite powers [8, Theorem 3.6]. Thus, the space X does not satisfy $\binom{\Omega}{\Gamma}$. □

5. More applications to function spaces

Corollary 13. *Let $T \subseteq [\mathbb{N}]^\infty$ be an unbounded tower of cardinality $\mathfrak{p}$.*

1. *For the real line topology, the space* $\mathrm{C}(T \cup \mathrm{Fin})$ *is Fréchet-Urysohn but the space* $\mathrm{B}(T \cup \mathrm{Fin})$ *is not strongly selectively separable.*
2. *Assume that $\aleph_1 < \mathfrak{p}$. For the Michael line topology, the space* $\mathrm{C}(T \cup \mathrm{Fin})$ *is strongly selectively separable and not Fréchet–Urysohn, and the space* $\mathrm{B}(T \cup \mathrm{Fin})$ *is not strongly selectively separable.* □

Corollary 14. *Let $X \subseteq \mathbb{R}$ be an uncountable set satisfying $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$. As a subset of $\mathbb{R}$, the space* $\mathrm{C}(-X \cup X)$ *is not Fréchet–Urysohn, but the space* $\mathrm{B}(-X \cup X)$ *is strongly selectively separable.* □

6. Remarks

Let X be a Tychonoff space. A set $U \subseteq X$ is a *co-zero* set if there is a continuous function $f\colon X \to \mathbb{R}$ such that $U = X \setminus f^{-1}(0)$. Let Ω_{cz} be the family of all *countable co-zero* ω-covers of the space. The space $\mathrm{C}(X)$ is strongly sequentially separable if and only if the space X satisfies $\binom{\Omega_{\mathrm{cz}}}{\Gamma}$ and has a coarser second countable topology [2, Theorem 54]. The space from Theorem **??**(2) has the property $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$, that is formally stronger than $\binom{\Omega_{\mathrm{cz}}}{\Gamma}$. In forthcoming Proposition 15, we show that the properties $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$ and $\binom{\Omega_{\mathrm{cz}}}{\Gamma}$ are different.

A family of sets is *almost disjoint* if the intersection of any two sets of this family is finite. For an almost disjoint family A in $[\mathbb{N}]^\infty$, let $\Psi(A)$ be the *Mrówka–Isbell space*, that is the set $A \cup \mathbb{N}$, where the points of $\mathbb{N}$ are isolated, and the neighborhoods of the points $a \in A$ are of the form $\{a\} \cup a \setminus b$, for sets $b \in \mathrm{Fin}$.

Proposition 15. *There is a maximal almost disjoint family A in $[\mathbb{N}]^\infty$ such that the Mrówka–Isbell space $\Psi(A)$ satisfies $\binom{\Omega_{\mathrm{cz}}}{\Gamma}$ but not $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$.*

Proof. There is a maximal almost disjoint family A in $[\mathbb{N}]^\infty$, of cardinality $|\mathbb{R}|$, such that the space $\Psi(A)$ satisfies $\binom{\Omega_{\mathrm{cz}}}{\Gamma}$ [2][Example 61, Theorem 54]. Let $A = \{ a_r : r \in \mathbb{R} \}$. Since $\mathbb{R}$ does not satisfy $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$, there is a family $\mathcal{U} \in \Omega_{\mathrm{ctbl}}(\mathbb{R})$ with no subfamily in $\Gamma(\mathbb{R})$. For each set $U \in \mathcal{U}$, let $U' := \{ a_r : r \in U \} \cup \mathbb{N}$. The family $\{ U' : U \in \mathcal{U} \}$ is in $\Omega_{\mathrm{ctbl}}(\Psi(A))$ and has no subfamily in $\Gamma(\Psi(A))$. Thus, the space $\Psi(A)$ does not satisfy $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$. □

A space is *projectively* $\binom{\Omega}{\Gamma}$ if, every continuous second countable image of that space, satisfies $\binom{\Omega}{\Gamma}$ [2].

Proposition 16. *For a Tychonoff space X, the following assertions are equivalent:*

1. *The space* $\mathrm{C}(X)$ *is strongly sequentially separable.*
2. *The space X has a coarser second countable topology, and it is projectively $\binom{\Omega}{\Gamma}$.*

Proof. (1) ⇒ (2): By the result of Gartside, Lo, and Marsh [6, Theorem 16], the space X has a coarser second countable topology. In order to prove that the space X is projectively $\binom{\Omega}{\Gamma}$, we show that it satisfies the equivalent property $\binom{\Omega_{\mathrm{cz}}}{\Gamma}$ [2, Theorem 54]. Let $F \subseteq \mathrm{C}(X)$ be a countable set such that the family $\mathcal{U} = \{ f^{-1}[\mathbb{R} \setminus \{0\}] : f \in F \}$ is an ω-cover of X. Let $\mathcal{B}$ be a countable basis of $\mathbb{R}$, and $\mathcal{B}'$ be a countable basis of a coarser topology on X. Let Y be the set X with the topology generated by the family $\{ f^{-1}[B] : B \in \mathcal{B} \} \cup \mathcal{B}'$. The space Y is second countable. By the result of Gartside, Lo, and Marsh [6, Theorem 16], the space Y satisfies $\binom{\Omega}{\Gamma}$. Since $\mathcal{U} \in \Omega(Y)$, the family $\mathcal{U}$ contains a cover $\mathcal{V} \in \Gamma(Y)$. Thus, $\mathcal{V} \in \Gamma(X)$.

(2) ⇒ (1): By (2), every coarser second countable topology for the space X satisfies $\binom{\Omega}{\Gamma}$. By the result of Gartside, Lo and Marsh [6, Theorem 16], the space $\mathrm{C}(X)$ is strongly sequentially separable. □

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